Four Dimensional Circulate Geometry and Lorentz Transformations

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Abstract. We found that the set of regular circulate matrices of order n is a group, giving rise to n dimensional circulate geometry. Some invariants are found. The question when Lorentz transformations are valid in it is of particular interest. We found the following facts: if $n = 2$ there exist the Lorentz transformations: if $n = 4$ there are double extended Lorentz transformations; if $n = 6$ there are triple extended Lorentz transformations. If $n = 3$ or $n = 5$ do not exist Lorentz transformations. The problem of arbitrary is still opened.

In the paper set of regular circulate matrices of order four is proven a group. The induced from this group geometry is called circulate geometry. We found some invariant. The main result was the fundamental formulas of this geometry preserving a pseudo-scalar product have double extended Lorenz transformations. We apply the computer algebras system of Maple 9.5. Without this system was impossible to investigate such problems.

1. Introduction

We use at first, the package `with (Linear Algebra):`
An arbitrary real circulate matrix of order four can be written in the form:
```
S: = Matrix ([[a, b, c, d], [d, a, b, c], [c, d, a, b], [b, c, d, a]]);
```

$$S := \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$$

If we write an other such matrices `m: = eval(S, [a = x,b = y,c = z,d = u]);`

$$m := \begin{bmatrix} x & y & z & u \\ u & x & y & z \\ z & u & x & y \\ y & z & u & x \end{bmatrix}$$

```
M: = eval(S, [a = X, b = Y, c = Z, d = U]);
```

$$M := \begin{bmatrix} X & Y & Z & U \\ U & X & Y & Z \\ Z & U & X & Y \\ Y & Z & U & X \end{bmatrix}$$

For the product `S.m` we have

$[a\,x+b\,u+c\,z+d\,y\,,a\,y+b\,x+c\,u+d\,z\,,a\,z+b\,y+c\,x+d\,u\,,$
$a\,u+b\,z+c\,y+d\,x]$
$[a\,u+b\,z+c\,y+d\,x\,,a\,x+b\,u+c\,z+d\,y\,,a\,y+b\,x+c\,u+d\,z\,,$
$a\,z+b\,y+c\,x+d\,u]$
$[a\,z+b\,y+c\,x+d\,u\,,a\,u+b\,z+c\,y+d\,x\,,a\,x+b\,u+c\,z+d\,y\,,$
$a\,y+b\,x+c\,u+d\,z]$
$[a\,y+b\,x+c\,u+d\,z\,,a\,z+b\,y+c\,x+d\,u\,,a\,u+b\,z+c\,y+d\,x\,,$
$a\,x+b\,u+c\,z+d\,y]$

Denoting **X:=(S.m)[1,1];Y:=(S.m)[1,2];Z:=(S.m)[1,3];U:=(S.m)[1,4];**

(1)
$$X := a\,x+b\,u+c\,z+d\,y$$
$$Y := a\,y+b\,x+c\,u+d\,z$$
$$Z := a\,z+b\,y+c\,x+d\,u$$
$$U := a\,u+b\,z+c\,y+d\,x$$

We get immediately **simplify (S. m - M);**

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, we established the product of two arbitrary circulate matrices is such matrix. For the next assertion is useful to know the following **factor (Determinant (M);**

$$-(X+U+Y+Z)\,(-X+Y-Z+U)\,(X^2-2\,X\,Z+Z^2-2\,Y\,U+U^2+Y^2)$$

Now we treat the inverse matrix of S: **i: = Matrix Inverse(S);**

$i :=$

$[(a^3-2\,a\,d\,b-c^2\,a+d^2\,c+c\,b^2)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2$
$+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)\,,-($
$b\,a^2-b^2\,d-2\,d\,c\,a+d^3+c^2\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2$
$+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)\,,-($
$c\,a^2-d^2\,a-a\,b^2-c^3+2\,c\,d\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2$
$+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)\,,-($
$d\,a^2-d^2\,b-2\,c\,b\,a+c^2\,d+b^3)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2$

$$+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)]$$

$$[-(d\,a^2-d^2\,b-2\,c\,b\,a+c^2\,d+b^3)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,(a^3-2\,a\,d\,b-c^2\,a+d^2\,c+c\,b^2)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,-(b\,a^2-b^2\,d-2\,d\,c\,a+d^3+c^2\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,-(c\,a^2-d^2\,a-a\,b^2-c^3+2\,c\,d\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)]$$

$$[-(c\,a^2-d^2\,a-a\,b^2-c^3+2\,c\,d\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,-(d\,a^2-d^2\,b-2\,c\,b\,a+c^2\,d+b^3)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,(a^3-2\,a\,d\,b-c^2\,a+d^2\,c+c\,b^2)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,-(b\,a^2-b^2\,d-2\,d\,c\,a+d^3+c^2\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)]$$

$$[-(b\,a^2-b^2\,d-2\,d\,c\,a+d^3+c^2\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,-(c\,a^2-d^2\,a-a\,b^2-c^3+2\,c\,d\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,-(d\,a^2-d^2\,b-2\,c\,b\,a+c^2\,d+b^3)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4),\,(a^3-2\,a\,d\,b-c^2\,a+d^2\,c+c\,b^2)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)]$$

Denoting: `a0:=i[1,1];b0:=i[1,2];c0:=i[1,3];d0:=i[1,4];`

$$a0 := (a^3-2\,a\,d\,b-c^2\,a+d^2\,c+c\,b^2)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)$$

$$b0 := -(b\,a^2-b^2\,d-2\,d\,c\,a+d^3+c^2\,b)\,/\,(a^4-4\,a^2\,d\,b-2\,a^2\,c^2+4\,a\,c\,b^2+4\,a\,d^2\,c+2\,d^2\,b^2-4\,c^2\,d\,b-d^4+c^4-b^4)$$

$$c0 := -(c\,a^2 - d^2\,a - a\,b^2 - c^3 + 2\,c\,d\,b) \big/ (a^4 - 4\,a^2\,d\,b - 2\,a^2\,c^2 + 4\,a\,c\,b^2 + 4\,a\,d^2\,c + 2\,d^2\,b^2 - 4\,c^2\,d\,b - d^4 + c^4 - b^4)$$

$$d0 := -(d\,a^2 - d^2\,b - 2\,c\,b\,a + c^2\,d + b^3) \big/ (a^4 - 4\,a^2\,d\,b - 2\,a^2\,c^2 + 4\,a\,c\,b^2 + 4\,a\,d^2\,c + 2\,d^2\,b^2 - 4\,c^2\,d\,b - d^4 + c^4 - b^4)$$

Introducing the circulate matrix `S0:= eval(S, [a=a0, b=b0, c=c0, d=d0]);`

$$S0 := \begin{bmatrix} a0 & b0 & c0 & d0 \\ d0 & a0 & b0 & c0 \\ c0 & d0 & a0 & b0 \\ b0 & c0 & d0 & a0 \end{bmatrix}$$

We get immediately `simplify (i-S0);`

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which shows the inverse matrix of any regular matrix of order 4 is such matrix. Thus, we have proved the following

Theorem 1 The set of regular circulate matrices of order 4 is a group. This group will be denoted by GC (4, R). The induced Klein geometry will be denoted by ΓC (4, R). Its transformation formulas are (1). Using these formulas, we get easy:

```
factor(X+U+Y+Z);
```

$$(x + u + y + z)\,(c + a + b + d)$$

```
factor(X-Y+Z-U);
```

$$-(-x + y - z + u)\,(c + a - b - d)$$

```
factor(X^2-2*X*Z+Z^2-2*Y*U+U^2+Y^2);
```

$$(x^2 + z^2 + u^2 + y^2 - 2\,x\,z - 2\,y\,u)\,(b^2 + c^2 + d^2 - 2\,c\,a + a^2 - 2\,b\,d)$$

Thus, we can formulate the following

Theorem 2 The following expressions are invariant:

$$J1 := \frac{X + U + Y + Z}{X1 + U1 + Y1 + Z1} = \frac{x + u + y + z}{x1 + u1 + y1 + z1}$$

$$J2 := \frac{X - Y + Z - U}{X1 - Y1 + Z1 - U1} = \frac{x - y + z - u}{x1 - y1 + z1 - u1}$$

$$J3 := \frac{X^2 - 2\,X\,Z + Z^2 - 2\,Y\,U + U^2 + Y^2}{X1^2 - 2\,X1\,Z1 + Z1^2 - 2\,Y1\,U1 + U1^2 + Y1^2} = \frac{x^2 - 2\,x\,z + z^2 - 2\,y\,u + y^2 + u^2}{x1^2 - 2\,x1\,z1 + z1^2 - 2\,z1\,u1 + u1^2 + y1^2}$$

Of any two vectors **(n, y, z, u), (x_1, y_1, z_1, u_1)**.

2. Lorentz Transformations We introduce the pseudo-scalar product:

```
f:=x1*x2+y1*y2+K*(z1*z2+u1*u2);
```

$$f := x1\,x2 + y1\,y2 + K\,(z1\,z2 + u1\,u2)$$

For the corresponding vectors, we have `F: =X1*X2+Y1*Y2+K*(Z1*Z2+U1*U2);`

$$F := X1\,X2 + Y1\,Y2 + K\,(Z1\,Z2 + U1\,U2)$$

Then we state the question when this product is invariant. We apply the transformation formulas (1) for both vectors:

$$(x_1, y_1, z_1, u_1), (x_2, y_2 m z_2, u_2)$$

For some special vectors we calculate:1.

1. `eval(f,[x1=1,x2=1,y1=0,y2=0,z1=0,z2=0,u1=0,u2=0]);`

$$1$$

`eval(F,[x1=1,x2=1,y1=0,y2=0,z1=0,z2=0,u1=0,u2=0]);`

$$a^2 + b^2 + K\,(c^2 + d^2)$$

2. `eval(f,[x1=0,x2=0,y1=1,y2=1,z1=0,z2=0,u1=0,u2=0]);`

`eval(F,[x1=0,x2=0,y1=1,y2=1,z1=0,z2=0,u1=0,u2=0]);`

$$1$$

$$d^2 + a^2 + K\,(b^2 + c^2)$$

3. `eval(f,[x1=0,x2=0,y1=0,y2=0,z1=1,z2=1,u1=0,u2=0]);`

`eval(F,[x1=0,x2=0,y1=0,y2=0,z1=1,z2=1,u1=0,u2=0]);`

$$K$$

$$c^2 + d^2 + K\,(a^2 + b^2)$$

4. `eval(f,[x1=0,x2=0,y1=0,y2=0,z1=0,z2=0,u1=1,u2=1]);`

`eval(F,[x1=0,x2=0,y1=0,y2=0,z1=0,z2=0,u1=1,u2=1]);`

$$K$$

$$b^2 + c^2 + K\,(d^2 + a^2)$$

5. `eval(f,[x1=0,x2=-1,y1=0,y2=0,z1=0,z2=0,u1=1,u2=1]);`

`eval(F,[x1=0,x2=-1,y1=0,y2=0,z1=0,z2=0,u1=1,u2=1]);`

$$\mathbf{K}$$
$$b\,(-a+b)+c\,(-b+c)+K\,(d\,(-c+d)+a\,(a-d))$$

6. `eval(f,[x1=0,x2=0,y1=0,y2=1,z1=0,z2=0,u1=1,u2=0]);`
`eval(F,[x1=0,x2=0,y1=0,y2=1,z1=0,z2=0,u1=1,u2=0]);`

$$\mathbf{0}$$
$$b\,d+c\,a+K\,(b\,d+c\,a)$$

7. `eval(f,[x1=0,x2=1,y1=-1,y2=1,z1=0,z2=-1,u1=0,u2=0]);`
`eval(F,[x1=0,x2=1,y1=-1,y2=1,z1=0,z2=-1,u1=0,u2=0]);`

$$\mathbf{-1}$$
$$-d\,(a-c+d)-a\,(a+b-d)+K\,(-b\,(-a+b+c)-c\,(-b+c+d))$$

We solve the system:

```
solve({K*d^2+a^2+b^2+K*c^2=1,K*b^2+a^2+d^2+K*c^2=1,
K*b^2+K*a^2+c^2+d^2=K,K*d^2-b*a+K*a^2-K*d*c-K*a*d-
c*b+b^2+c^2=K,c*a+b*d+K*b*d+K*c*a=0,-K*b^2-b*a+d*c+K*b*a-K*d*c-
a^2-d^2-K*c^2=-1},{a,b,c,d,K});
```

$$\{d=\mathrm{RootOf}(\_Z^2+c+c^2),\,K=1,\,a=-1-c,\,b=-\mathrm{RootOf}(\_Z^2+c+c^2),$$
$$c=c\},\,\{K=1,\,a=1-c,\,d=\mathrm{RootOf}(\_Z^2-c+c^2),$$
$$b=-\mathrm{RootOf}(\_Z^2-c+c^2),\,c=c\},\,\{b=-d-1,$$
$$a=-\mathrm{RootOf}(d^2+d+\_Z^2),\,K=1,\,c=\mathrm{RootOf}(d^2+d+\_Z^2),\,d=d\},$$
$$\{b=-d+1,\,K=1,\,a=-\mathrm{RootOf}(d^2-d+\_Z^2),$$
$$c=\mathrm{RootOf}(d^2-d+\_Z^2),\,d=d\},\,\{c=\frac{1}{2},\,b=\frac{1}{2},\,K=1,\,d=\frac{1}{2},\,a=\frac{-1}{2}\},$$
$$\{c=\frac{1}{2},\,K=1,\,d=\frac{1}{2},\,a=\frac{1}{2},\,b=\frac{-1}{2}\},$$
$$\{b=\frac{1}{2},\,K=1,\,a=\frac{-1}{2},\,d=\frac{-1}{2},\,c=\frac{-1}{2}\},\qquad \{c=\frac{1}{2},\,b=\frac{1}{2},\,K=1,\,a=\frac{1}{2},\,d=\frac{-1}{2}\},$$
$$\{K=1,\,a=\frac{1}{2},\,d=\frac{-1}{2},\,b=\frac{-1}{2},\,c=\frac{-1}{2}\},\qquad \{c=\frac{1}{2},\,K=1,\,a=\frac{-1}{2},\,d=\frac{-1}{2},\,b=\frac{-1}{2}\},$$
$$\{b=\frac{1}{2},\,K=1,\,d=\frac{1}{2},\,a=\frac{1}{2},\,c=\frac{-1}{2}\},$$
$$\{K=1,\,d=\frac{1}{2},\,a=\frac{-1}{2},\,b=\frac{-1}{2},\,c=\frac{-1}{2}\},$$
$$\{b=0,\,d=0,\,K=K,\,c=0,\,a=1\},$$

$$\{b = 0, a = -1, d = 0, K = K, c = 0\},$$
$$\{b = 0, a = \mathrm{RootOf}(-1 + \_Z^2 - c^2), K = -1, d = 0, c = c\}$$

We see, two types of these solutions are interested: One solution – the last one:

```
d = 0, a = RootOf(-1+_Z^2-c^2), K = -1, b = 0, c = c
```

Denote: `d1:=0; b1:=0; a1:=sqrt (1+c^2); K1:=-1;`

$$d1 := 0$$
$$b1 := 0$$
$$a1 := \sqrt{1+c^2}$$
$$K1 := -1$$

We calculate `S1:=eval(S, [a=a1, b=b1, d=d1]);`

$$S1 := \begin{bmatrix} \sqrt{1+c^2} & 0 & c & 0 \\ 0 & \sqrt{1+c^2} & 0 & c \\ c & 0 & \sqrt{1+c^2} & 0 \\ 0 & c & 0 & \sqrt{1+c^2} \end{bmatrix}$$

`Is Orthogonal (S1);` **false**

`Simplify (eval (F, [a=a1, b=b1, d=d1, K=-1]));`

$$x1\,x2 + y1\,y2 - z1\,z2 - u1\,u2$$

Second solution: `K = 1, d = RootOf (_Z^2-c+c^2),`

`b = -RootOf(_Z^2-c+c^2), a = -c+1, c = c`

We put: `d2:=sqrt(c-c^2); a2:=1-c; b2:=-d2;`

$$d2 := \sqrt{c-c^2}$$
$$a2 := 1-c$$
$$b2 := -\sqrt{c-c^2}$$

And calculate `S2:=eval(S, [a=a2, d=d2, b=b2]);`

$$S2 := \begin{bmatrix} 1-c & -\sqrt{c-c^2} & c & \sqrt{c-c^2} \\ \sqrt{c-c^2} & 1-c & -\sqrt{c-c^2} & c \\ c & \sqrt{c-c^2} & 1-c & -\sqrt{c-c^2} \\ -\sqrt{c-c^2} & c & \sqrt{c-c^2} & 1-c \end{bmatrix}$$

`Is Orthogonal (S2);` ***true***

```
Simplify (eval (F, [a=a2, d=d2, b=b2, K=1]));
```

$$z1\ z2 + y1\ y2 + x1\ x2 + u1\ u2$$

Now we investigate the formulas (1) under the matrix S_1:

```
X0:=eval(X,[a=a1,b=b1,d=d1]);Y0:=eval(Y,[a=a1,b=b1,d=d1]);Z0:=ev
al(Z,[a=a1,b=b1,d=d1]);U0:=eval(U,[a=a1,b=b1,d=d1]);
```

$$X0 := \sqrt{1+c^2}\ x + c\ z$$

$$Y0 := \sqrt{1+c^2}\ y + c\ u$$

$$Z0 := \sqrt{1+c^2}\ z + c\ x$$

$$U0 := \sqrt{1+c^2}\ u + c\ y$$

We put and calculate: `c: =sinh (t); simplify (a1);`

$$c := \sinh(t)$$

$$\operatorname{csgn}(\cosh(t))\cosh(t)$$

Then we get `X:=cosh(t)*x+sinh(t)*z;Z:=sinh(t)*x+cosh(t)*z;`

$$X := \cosh(t)\,x + \sinh(t)\,z$$

$$Z := \sinh(t)\,x + \cosh(t)\,z$$

```
Y: =cosh(t)*y+sinh(t)*u; U:=sinh(t)*y+cosh(t)*u;
```

$$Y := \cosh(t)\,y + \sinh(t)\,u$$

$$U := \sinh(t)\,y + \cosh(t)\,u$$

Thus, we prove the following

Theorem 3 The circulate transformations (1) preserve the pseudo-scalar product if they are

$$X := \cosh(t)\,x + \sinh(t)\,z$$

$$Z := \sinh(t)\,x + \cosh(t)\,z$$

$$Y := \cosh(t)\,y + \sinh(t)\,u$$

$$U := \sinh(t)\,y + \cosh(t)\,u$$

- The double extended Lorentz transformations.

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