Lecture Notes

on Conformal Field Theory

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Content

1. Introduction, conformal transformations, Virasoro algebra, Verma modules

1.1. Why 2d conformal field theory?

Basically:

• Strings - CFT enters this subject in many ways, and in fact has its origins in the string theory, or the "dual theory". To start with, when propagating in time strings are objects that sweep out a 2-dim surface, their world sheet and the proper description of the strings is in terms of fields on this surface, i.e., as 2d fields.

• CFT describes 2d critical phenomena, i.e, the behaviour of some models of statistical physics in the vicinity of critical points, where typically the correlation length (inverse of mass), as a function of some of the parameters like temperature, becomes divergent, i.e. the theory becomes approximately massless. In that case the exponential decay of the correlation functions is replaced by a power low, which is scaling invariant . The exponents of such law are part of the data which the CFT provides in a systematic way. Diffferent models may lead to the same data - that is why CFT can classify the universality classes of statistical models.

• Finally - 2d CFT provided first examples of exactly solvable field theoretic models, the reason being that it exploits a huge symmetry, described by infinite dimensional algebras; so it is looked as laboratory of methods, some of which may be useful in higher dimensional theory. Furthermore the 2d CFT exploits and makes many connections with various branches of mathematics.

The study of CFT goes - in two steps.

Characteristic feature of the 2d theory is the chiral factorisations into holomorphic and anti-holomorphic parts. Hence one can study a lot of properties just of the one of these 1d structures. The correlators are then holomorphic functions and one can apply complex analysis for their study. At a second step one has to impose locality to describe the full theory, the physical fields.

Literature:

• The modern development of the 2d CFT started with the fundamental paper (1984) by A. Belavin A.Polyakov and A. Zamolodchikov [1]. They were the first to exploit the full power of the representation theory of an infinite dimensional algebra, the Virasoro algebra, previously developed by the mathematicians (Kac, Feigin and Fuks). BPZ combined it with the field theory of local fields so that the infinite symmetry plus associativity of operator product expansion (OPE) describes all correlation functions; this, so called, "bootstrap approach" does not assume an existence of lagrangian.

• Basic textbook to be followed in these lectures is the book on CFT by P. Di Francesco, P. Mathieu and D. Sénéchal [2]. In this book there are also chapters on statistical mechanics, lattice models, critical phenomena;

- other reviews on CFT, e.g., [3], [4], [5], [6], [7], [8];
- string theory - any review contains chapters on CFT methods and applications;
- more recent - review on applications to topological quantum computation [9].

1.2. Conformal transformations

A coordinate transformations $x^\mu \to x^{'\mu}$ that leaves invariant the metric up to an overall factor

$$g^{'}_{\alpha\beta}(x') = \omega^2(x) g_{\mu\nu} = \omega^2(x)\eta_{\mu\nu}$$

is called conformal. In other words these are the transformations that preserve the angles $\cos\theta = \frac{a\cdot b}{\sqrt{a^2 b^2}}$ between two vectors, $a \cdot b = g_{\mu\nu} a^\mu b^\nu$; obviously translations and rotations - form a subgroup since they preserve the metric.

Consider a pseudo-euclidean space, i.e., $\mathbb{R}^d$ with a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ of signature $(d-m, m)$ and line element

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu$$

Under a change of coordinates

$$x^\mu \to x^{'\mu} \sim x^\mu + \zeta^\mu(x) \to \frac{\partial x^\mu}{\partial x^\alpha} \sim \delta^\mu_\alpha - \partial^\mu \zeta_\alpha$$

the metric transforms as

$$g^{'}_{\alpha\beta}(x') = \eta_{\mu\nu}\,(x) \frac{\partial x^\mu}{\partial x'^{\,\alpha}} \, \frac{\partial x^\nu}{\partial x'^{\,\beta}} \sim \eta_{\alpha\beta} - \partial_\alpha \zeta_\beta - \partial_\beta \zeta_\alpha$$

$$g^{'}_{\alpha\beta}(x') = \omega^2(x) g_{\mu\nu} = \omega^2(x)\eta_{\mu\nu} \sim (1-f)\eta_{\alpha\beta}$$

Infinitesimally (linearly in ζ_α) we obtain the differential eqs

$$\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha = \frac{2}{d} \partial_\mu \zeta^\mu \; \eta_{\alpha\beta} \, , \tag{1.1}$$

The solution for $d \neq 1, 2$ is at most quadratic polynomial; main steps in the proof: for $d > 1$ one derives from these eqs

$$\begin{aligned} &1.\ 2\partial_\mu\partial_\nu\zeta_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \\ &2.\ (2-d)\partial_\mu\partial_\nu\partial\cdot\zeta = 0 \\ &\text{from 1.} \Rightarrow \partial\cdot\zeta \text{ is at most linear for } d \neq 2, \text{inserting in 2.} \Rightarrow \\ &\partial_\mu\partial_n\zeta_\rho = \text{ const, hence } \zeta_\rho - \text{ quadratic} \end{aligned} \tag{1.2}$$

The solution reads - with small constants $\epsilon^a = (a^\mu, b^\mu, \omega^{\mu\nu} = -\omega^{\nu\mu}, c)$

$$\zeta^\alpha = -a^\mu\delta^\alpha_\mu - \omega^{\mu\nu}(x_\mu\delta^\alpha_\nu - x_\nu\delta^\alpha_\mu) - cx^\alpha - b^\mu(x^2\delta^\alpha_\mu - 2x_\mu x^\alpha) = \sum_a \epsilon^a\bar{\zeta}^\alpha_a$$

This defines generators of 1-parameter transformations (conformal Killing vectors)

$$\zeta^\alpha\partial_\alpha = \sum_b \epsilon^b X_b\,, \quad X_a = \sum_\alpha \bar{\zeta}^\alpha_a\partial_\alpha[= \partial_{\epsilon^a}F(x + \epsilon^a\bar{\zeta}_a)_{|\epsilon^a=0}]$$

(no summation in a in the last expression). Altogether we have $2d + 1 + d(d-1)/2 = (d+2)(d+1)/2$ - range of the index a

$$\begin{aligned} &\bar{\zeta}^\alpha_\mu = -\delta^\alpha_\mu \text{ (infinites. transl. } - \text{only one of the coordinates is shifted), generator} \\ &T_\mu = -iP_\mu = X_{\mu 2h+1} + X_{\mu 2h+2} = \bar{\zeta}^\alpha_\mu\partial_\alpha = -\partial_\mu \\ &\text{(infinites. Lorentz transformations), } X_{\mu\nu} = -iM_{\mu\nu} = \bar{\zeta}^\alpha_{\mu\nu}\partial_\alpha = x_\nu\partial_\mu - x_\mu\partial_\nu \\ &\text{(inf. dilatations)}\,,\ X_{2h+1,2h+2} = -x\partial \\ &\text{(inf. special conformal transformations)}\,, X_{\mu 2h+2} - X_{\mu 2h+1} = 2x_\mu x\partial - x^2\partial_\mu \end{aligned} \tag{1.3}$$

algebra of $d{+}2 \times d{+}2$ real matrices skew-symmetric X_{AB} - the Lie algebra of the connected components of the pseudo-orthogonal group $SO_0(d-m+1, m+1)$ in $d+2$ dims;

$$[X_{AB}, X_{CD}] = \eta_{AC}X_{BD} + \eta_{BD}X_{AC} - \eta_{AD}X_{BC} - \eta_{BC}X_{AD}$$

with matrix realisation $(X_{AB})^C_D = \eta_{AD}\delta^C_B - \eta_{bD}\delta^C_A$.

Generate the global tranformations - exponentiating

$$\begin{aligned} &x^{'\alpha} = x^\alpha + a^\alpha \\ &x^{'\alpha} = \Lambda^\alpha_\nu x^\nu \sim x^\alpha + \omega^{\mu\nu}(x_\mu\delta^\alpha_\nu - x_\nu\delta^\alpha_\mu)\,, \Lambda \in SO(d-m, m) \\ &x^{'\alpha} = \rho x^\alpha \sim (1+a)x^\alpha \\ &x^{'\alpha} = \frac{x^\alpha - b^\alpha x^2}{1 - 2b\cdot x + b^2x^2} \sim x^\alpha + 2x^\alpha b\cdot x - b^\alpha x^2 \end{aligned} \tag{1.4}$$

$\omega = 1$ for the Poincare subgroup; transformation - isometry, while $\omega = \rho^{-1}$ for dilatations and $\omega = 1 - 2b \cdot x + b^2 x^2$ for special conf transf. The conformal group (of M) is locally isomorphic to the identity component $SO_0(d-m+1, m+1)$ of the pseudo-orthogonal group in $d+2$ dims.; acts properly on a compactified space $\bar{M}$ - "adding singularities"

In 2d (1.4) gives only part of the solutions. In euclidean metric $\eta_{\mu\nu} = \delta_{\mu\nu}$ the subset of these solutions form the 6-dim group $SO_e(3,1)$ - identity component of the Lorentz group, $SO_e(3,1)$ - isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$

Now there are two independent equations for two functions - the eqs become the Cauchy-Riemann equations for holomorphic functions

$$\begin{aligned} &\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \,, \ \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \\ &\Rightarrow \partial_z \epsilon^{\bar{z}} \pm \partial_{\bar{z}} \epsilon^z = 0 \,, \ \partial_z \epsilon^{\bar{z}} = 0 = \partial_{\bar{z}} \epsilon^z \end{aligned} \tag{1.5}$$

where complex coordinates were introduced $z = x^1 + ix^2$ and $\epsilon^z = \epsilon^1 + i\epsilon^2$. In other words, $\epsilon = \epsilon^z$ is a holomorphic function of z and $\bar{\epsilon} = \epsilon^{\bar{z}}$ - anti-holomorphic. The group of conformal transformations is therefore isomorphic to the infinite dimensional group of arbitrary analytic coordinate transformations.

In complex coordinates line element is

$$ds^2 = dz\, d\bar{z} = \frac{1}{2} dz\ d\bar{z} + \frac{1}{2} d\bar{z}\ dz = g_{z\bar{z}} dz\ d\bar{z} + g_{\bar{z}z} d\bar{z}\bar{z}$$

i.e., the euclidean metric reads $g_{zz} = 0 = g_{\bar{z}\bar{z}}$, $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$, hence $g^{z\bar{z}} = 2$.

Expanding

$$\epsilon(z) = -\sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}$$

in Laurent series we get the generators

$$\epsilon(z)\partial_z = -\sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \partial_z = \sum_{n\in\mathbb{Z}} \epsilon_n l_n \,,$$

and same for the complex conjugated; each satisfy the *Witt algebra*,

$$[l_n, l_m] = (n-m) l_{n+m}$$

l_n and $\bar{l}_n$ commute. I.e., the conformal algebra splits into a direct sum of two algebras $\mathcal{A} \oplus \mathcal{A}$; we can treat in most problems z and $\bar{z}$ as independent, not complex conjugate, i.e.,

extend the real components x^μ to complex values, $\mathbb{R}^2 \to \mathbb{C}^2$; physical space back $\bar{z} = z^*$.
NB: for $z = -1/w$ investigate the behaviour at infinity transforming the vector field

$$\epsilon(z)\partial_z = -\sum_{n\in\mathbb{Z}} \epsilon_n w^{-n+1}\partial_w$$

these (infinitesimal) transformations of the complex plane $\mathcal{C}$ are singular at 0 or ∞, except for $n = 0, \pm 1$. Extending to global transformations - holomorphic maps (local, on some domain of the Riemann sphere $S^2 = \mathbb{C} \cup \infty$). Globally defined (finite) invertible holomorphic maps on the Riemann sphere are only the projective transformations $SL(2, \mathcal{C})/\mathbb{Z}_2$, the group of two by two complex matrices with unit determinant

$$\begin{aligned} w(z) &= \frac{az+b}{cz+d}\,, \text{ or, } g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, z = z_1/z_2 \\ g &= \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1/\sqrt{\rho} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \end{aligned} \tag{1.6}$$

translations, rotations, dilatations, special conformal transformations; generators $sl(2, \mathcal{C})$: $l_n + \bar{l}_n\,, i(l_n - \bar{l}_n)\,, n = -1$ (translations), $n = 1$ (special conformal transformations, $n = 0$ dilatations and rotations resp.

$$M = i(l_0 - \bar{l}_0) = x_\mu\partial_\nu - x_\nu\partial_\mu\,,\ D = \ l_0 + \bar{l}_0 = x_\mu\partial_\mu\,;$$

- In the quantum field theory the symmetry with respect to the finite dimensional subalgebra determines in arbitrary dimensions the (scalar in $d > 2$) 3-point functions up to arbitrary constant and restricts the functional dependence of the n-point functions. In 2d (or 1d) - infinite group - we expect more restrictions on the correlators. However it turns out that few of the representations of the Witt algebra are non-trivial and interesting - rather one has to extend it $\to$ central extension $\to$ Virasoro algebra.

1.3. Virasoro algebra, Verma modules

Infinite dimensional Lie algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + c\frac{n(n^2-1)}{12}\ \delta_{n+m,0}\,,\quad [L_n, c] = 0 \tag{1.7}$$

central extension of Witt algebra.

- Verma modules - highest (lowest) weight representations:

$$L_n|h\rangle = 0 \text{ for } n > 0\,,\ L_0|h\rangle = h|h\rangle\,,$$
$$V(h,c) = \text{linear span of } \{L_{-n_1}\,L_{-n_2}\,\ldots L_{-n_k}|h\rangle\,,\ 1 \leq n_1 \leq n_2 \leq n_k\,,\ k \geq 1\} \tag{1.8}$$

$$L_0\ v_{n,\{n_s\}} = (h+n)v_{n,\{n_s\}}\,,\ n = \sum_{s=1}^{k} n_s$$

for any non-negative integer n there is a finite subspace of eigenstates $v_{n,\{n_s\}}$ of L_0 and $V(h,c)$ is an infinite direct sum over such subspaces of fixed eigenvalue $h+n$ of L_0 - $\mathbb{Z}_+$-graded module.

- For the fixed positive integer n the number of distinct linearly independent (descendent) states $= p(n)$ - partition function =number of partitions of n - $(p(1), p(2), p(3), p(4), p(5)...) = (1, 2, 3, 5, 7, ...)$, etc., with generating function

$$\frac{1}{\varphi(q)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{k=1} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} q^{s_1+2s_2+\ldots ks_k} = \sum_{n=0}^{\infty} p(n)\, q^n \tag{1.9}$$

φ - Euler function.

The function (1.9) of an indeterminate variable q defines the formal character of the Verma module $V(h,c)$ - counts multiplicities of L_0 eigenvalue subspaces:

$$\chi_{V(h,c)} = q^{-\frac{c}{24}} \sum_{n=0}^{\infty} \text{mult}_{V(h,c)}(n) q^{h+n} = \frac{q^{h-\frac{c}{24}}}{\varphi(q)} = tr_{V(h,c)} q^{L_0 - \frac{c}{24}} \tag{1.10}$$

- The last representation in (1.10) requires a norm - introduce a hermitian form

  $L_n^+ = L_{-n}\,,\ \langle v_0|L_{-n} = 0\,,\ n > 0\,,\ |v_0|^2 = \langle v_0|v_0\rangle = 1$

  $\Rightarrow \langle v_n|v_m\rangle = 0$ if $n \neq m$.

- Reducible Verma modules - there exist singular vectors - higher grade "vacuum" states $|h'\rangle,\ L_n|h'\rangle = 0$ for $n > 0 \ \Rightarrow$ deneracy of form $\langle v|h'\rangle = 0$ for any state v of $V(h,c)$; singular vector - null state.

Singular vectors generate Verma submodules; all states in this submodule has zero norm. The factor over the union of such submodules is isomorphic to an irreducible module.

Examples: level 1:

$$|h+1\rangle = L_{-1}|h\rangle$$

is a singular vector iff $h = 0$, while c - arbitrary. If c is generic (non-rational) - this is the only singular vector in $V(0,c)$ and the factor $L_{h=0} = V(0,c)/V(1,c)$ defines an irreducible module; character - subtract states of the submodule

$$\chi_{L_0} = \chi_{V(0,c)} - \chi_{V(1,c)} = q^{-\frac{c}{24}} \frac{1-q}{\varphi(q)} \tag{1.11}$$

NB: The h.w. state $|0\rangle$ of the irreducible factor-module is annihilated by $L_n\,,\ n \geq -1$.

Ex. : compare with l.w. Verma modules M_{-j} of the finite dimensional subalgebra $sl(2)$ generated by $\{L_0, L_1, L_{-1} \to (\frac{1}{2}H, -E^+, -E^-)$,

$$\{v_n^{(j)} = L_{-1}^n |h = -j\rangle\}\,, \quad L_1 v_n^{(j)} = n(n-1-2j) v_{n-1}^{(j)}$$

hence reducible for $2j+1 \in \mathbb{Z}_{>0}$, singular vector $v_{2j+1}^{(j)} = L_{-1}^{2j+1} v_{-j}$ of h.w. $n - j = j+1$

$$\begin{array}{cccccccc} \bullet & - - - & \bullet & - - - & \bullet & - - - - - - - - - - - - & \bullet & - - - - - - \\ -j & & -j+1 & & & & j+1 & \end{array} \tag{1.12}$$

factor representations M_{-j}/M_{j+1} - finite, $(2j+1)$ dimensional, with formal characters

$$\chi_j = \frac{q^{-j}}{1-q} - \frac{q^{j+1}}{1-q} = \sum_{p=-j}^{j} q^p$$

Recall: evaluating $q = e^{2\pi i \phi}$ we get the characters of $SU(2)$,

$$\chi_j(\phi) = tr_j(e^{2i\phi L_0}) = \frac{\sin(2j+1)\phi}{\sin\phi} \tag{1.13}$$

- There exists an analogous series of reducible Vir Verma modules parametrised with a ("quantised isospin")

$$h_j = -j + j(j+1)t\,, \quad 2j \in \mathbb{Z}_+\,, t \in \mathbb{C}$$

a singular vector appears at $h' = h_{-j-1} = (2j+1) + h_j$ (the only one for $t \neq \mathbb{Q}$). The simplest non-trivial example, $j = 1/2$ - at level 2.

Literature: [2].

2. Reducibility, singular vectors, minimal representations

We continue with another example of a singular vector at level 2.

Level 2:

$v_2 = (L_{-1}^2 - tL_{-2})|h\rangle$

sufficient to compute the action of $L_1 \,, L_2$,

$$\begin{aligned} &\text{from } L_1 : \to v_2 = (L_{-1}^2 - \frac{2(2h+1)}{3} L_{-2})|h\rangle \\ &\text{from } L_2 : \to v_2 = (L_{-1}^2 - tL_{-2})|h\rangle \,, \quad t^2 + \frac{c-13}{6} t + 1 = 0 \,, \end{aligned} \tag{2.1}$$

two solutions $\big($one for $c = 1, c = 25)$

$$\Rightarrow \quad h = \tfrac{1}{2}(\tfrac{1}{2}+1)\,t - \frac{1}{2} \,, \quad t = \frac{1}{12}(13 - c \pm \sqrt{(1-c)(25-c)}) \tag{2.2}$$

NB: $t > 0$ for for $c \leq 1$, $t < 0$ for $c \geq 25$, and $|t| = 1$ for $1 < c < 25$.

Choosing one of the roots, we will denote $h_{2,1} = \frac{3t}{4} - 1/2 \,, h_{1,2} = \frac{3}{4t} - 1/2$

Alternatively we can compute (Kac) determinant of the matrix of scalar products $\langle i|j\rangle = \langle j|i\rangle$ of basic states at a given level, e.g., at level 2 we have to compute

$M_{11}^{(2)} = \langle h|L_1^2 L_{-1}^2|h\rangle \,, M_{12}^{(2)} = M_{21}^{(2)} = \langle h|L_1^2 L_{-2}|h\rangle \,, M_{22}^{(2)} = \langle h|L_2 L_{-2} h\rangle$, so that

$$\begin{aligned} M^{(2)} &= \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + \frac{c}{2} \end{pmatrix} \,, \\ \det M^{2)} &= (h - h_{1,1})(h - h_{2,1})(h - h_{1,2}) \end{aligned} \tag{2.3}$$

The three roots of $\det M^{2)}$ correspond to the three null states representing the singular vectors up to level 2.

• Generally, the reducible Vir Verma modules $V(h_{r,r'}, c)$ are parametrised by two positive integers

$$h_{r,r'} = (r - \frac{r'}{t})^2 \frac{t}{4} + \frac{c-1}{24} \,, \quad r, r \in \mathbb{Z}_{>0} \,, \; c = 13 - 6(t + \tfrac{1}{t}) \tag{2.4}$$

also convenient parametrisation

$$h_J = J(J+1)\,t - J \,, \; J = j - j'/t \,, \; 2j, 2j' \in \mathbb{Z}_+ \tag{2.5}$$

The Verma module $V(h_{r,r'}, c)$ has a singular vector at level

$$h_{r,r'} + rr' = h_{-r,r'} = h_{r,-r'} \tag{2.6}$$

1. For $t \neq \mathbb{Q}$ this is the only singular vector
2. For $t \in \mathbb{Q}$ integers (possible for $c \leq 1$ or $c \geq 25$):

- case $c < 1$: $t = p'/p\,, p, p' \geq 2$ - positive coprime integers - each of the Verma modules (2.4) has an infinite number of singular vectors, labelled by a h.w. in the set (2.4); we shall use in this case also the parametrisation $t = b^2$ with real b and for the scaling dims

$$\begin{aligned} c = 13 - 6(b^2 + 1/b^2) = 1 - 6\alpha_0^2\,, \quad \alpha_0 = \tfrac{1}{b} - b; \\ h(\alpha) = \alpha(\alpha - \alpha_0)\,, \; \alpha = \alpha_{r,r'} = \tfrac{r-1}{2}b - \tfrac{r'-1}{2b} = Jb\,, \end{aligned} \tag{2.7}$$

while the formula (2.4) becomes

$$h_{r,r'} = \frac{(p'r - r'p)^2 - (p-p')^2}{4pp'}$$

Of main interest in this case are the Verma modules with r, r' restricted by (Kac table)

$$\begin{aligned} 1 \leq r = 2j+1 \leq p-1\,, \quad 1 \leq r' = 2j+1 \leq p'-1\,, \\ \text{factorised by } \mathbb{Z}_2 \text{ symmetry } : h_{r,r'} = h_{p-r,p'-r'} \end{aligned} \tag{2.8}$$

Because of the general relation $h_{r,r'} = h_{r+kp,r'+kp'}$ (we have also $h_{r,r'} = h_{-r,-r'}$, and then combinations) the module has infinitely many singular vectors. Let us describe the pattern of embeddings : besides the singular vector (2.6) at level rr' with dimension $h_{-r,r'} = h_{p+r,p'-r'}$, it has a second singular vector at level $(p-r)(p-r')$ with $h_{r-p,p'-r'} = h_{2p-r,r'}$; each of them is again reducible being again represented by positive integers etc - infinite pattern of embedings.

$$V(h_{r,r'}, c) \supset V(h_{-r,r'}, c) \cup V(h_{2p-r,r'}, c) \supset ....$$

Let us define two maps on the weights, $s_1^2 = 1 = s_0^2$

$$\begin{aligned} s_1 \cdot h_{r,s} = h_{-r,s}\,, \\ s_0 \cdot h_{r,s} = h_{2p-r,s}\,, \end{aligned} \tag{2.9}$$

Then the points on the upper line are parametrised by the sequence acting on $h_{r,s}$ $s_1, s_0s_1, s_1s_0s_1, ...$ while on the low line - by $s_0, s_1s_0, s_0s_1s_0, ...$;

$$
\begin{array}{ccccccc}
 & & & s_0 & & s_1 & \\
 & & (-r,s) & \longrightarrow & (2p+r,s) & \longrightarrow & \dots \\
 & s_1 \nearrow & & \searrow & & \searrow & \\
(r,s) & & & & & & \\
 & s_0 \searrow & & \nearrow & & \nearrow & \\
 & & (2p-r,s) & \longrightarrow & (r-2p,s) & \longrightarrow & \dots \\
 & & & s_1 & & s_0 &
\end{array}
\tag{2.10}
$$

The labels of the crossing arrows are recovered as compositions, e.g. the first SE corresponds to $s_1 s_0 s_1$, while the NE one, crossing it, to $s_0 s_1 s_0$. As explained, all the weights can be rewritten by positive integers, changing both (r, s).

NB: The rational $c > 25$ case is different - each reducible module is embedded in an infinite number of reducible Verma modules (the first one is irreducible). It is convenient to parametrise the case $c > 25$ with the same real b, but

$$
\begin{gathered}
c = 13 + 6(b^2 + 1/b^2) = 1 + 6Q^2\,,\ \ Q = \tfrac{1}{b} + b; \\
h(\beta) = \beta(Q-\beta)\,, \beta = \beta_{r,r'} = -\tfrac{r-1}{2}b - \tfrac{r'-1}{2b} = -Jb\,,\ \ 2j, 2j' \in \mathbb{Z}_+
\end{gathered}
\tag{2.11}
$$

the rational $c > 25$ case is now again parametrised by two positive coprime integers - $b^2 = p'/p$.

Furthermore the $c < 1$ diagrams reduce to a line for $t = 1/p$ or $t = p$, in particular $t = b = 1 = c$ - analogously a line for $c \geq 25$, i.e., negative such values.

- An irrep $M(h_{r,r'}, c_{p,p'})$ in the rational case is obtained by factoring over the states of this max submodule generated by the first two singular vectors,

$$
M(h_{r,r'}, c_{p,p'}) = V/(V(h_{-r,r'}, c) \cup V(h_{2p-r,r'}, c))
$$

On the other hand in counting just the multiplicities (not distinguishing the states) we have to account for the fact that we overcount some states belonging to both submodules, so we have to subtract them (that is add them to the mult in the initial module) - since these modules are themselves reducible, those overcounted states actually appear in their submodules, etc.; we get for the characters an infinite alternating sign sum of Verma module characters.

$$\chi_{M(h_{r,s})} = \chi_{h_{r,s}} + \sum_{k=1}(\chi_{(s_0 s_1)^k \cdot h_{r,s}} + \chi_{(s_1 s_0)^k \cdot h_{r,s}}) - \sum_{k=0}(\chi_{(s_0 s_1)^k s_0 \cdot h_{r,s}} + \chi_{(s_1 s_0)^k s_1 \cdot h_{r,s}})$$
$$= \sum_w \mathrm{sign(w)} \chi_{w \cdot h_{r,s}} \quad \text{or,}$$
$$\chi_{M(h_{r,s})} = K^{(p,p')}_{r,s} - K^{(p,p')}_{-r,s} \,, \qquad K^{(p,p')}_{r,s} := \frac{q^{-1/24}}{\phi(q)} \sum_{n\in\mathbb{Z}} q^{(2pp'n + pr' - p'r)^2/4pp'} \tag{2.12}$$

- The irreducible factors - **minimal series** of Vir modules - examples of **rational CFT** - a finite set for a fixed value of c .

- Unitarity

  Necessary conditions for positivity of the inner product - from

$$|L_{-1}v|^2 = 2h \Rightarrow \; h > 0$$

NB: The equality $h = 0$ does not appear here since the state $L_{-1}|0\rangle$ is removed and does not appear in the irreducible representation; the vacuum $|h = 0\rangle$ itself has norm 1.

$$|L_{-n}v_h|^2 = (2nh + (n^3 - n)\,\frac{c}{12})|v_h|^2 \,,$$

negative for sufficiently large n if $c < 0$, hence $c > 0$; in fact sufficient to analyse the identity rep

$$|L_{-n}v_0|^2 = (n^3 - n)\,\frac{c}{12}\,, \; n > 1 \,, \Rightarrow c > 0$$

The full analysis shows that the matrix $M^{(l)}$ is positive definite for $c \geq 1$ with no further restrictions on the dimension h. In the remaining range $0 < c < 1$ the only unitary representations - the particular subseries of the minimal set (2.8), $p' = p \pm 1$,

$$c = 1 - \frac{6}{m(m+1)} \,, m \geq 3$$

Literature: [10], [11], [1], [12], [13].

3. Virasoro primary fields, constraints from decoupling of singular vectors

3.1. Field realisation of the Vir representations

There is an infinite family $[\phi_h(z)]$ of fields associated with a Verma module, or an irreducible factor- module. The fields associated with the vacuum (l.w.) state of Vir irreducible modules are called *primary fields.* Identity primary field - $v_0 = |0\rangle$, and

$$\phi_h(0)|0\rangle = |h\rangle \,, \langle h| = \lim_{z\to\infty} z^{2h}\langle 0|\phi_h(z)$$

Interpretation: $z = e^{\tau+i\sigma}$, $|\sigma| \leq \pi \,, \tau$ - "time", the two vacuum states correspond to $\tau \to \mp\infty$; radial quantisation - fixed "time" circles on the cylinder - closed contours in the plane; radial direction in the plane - direction of "time"

$$[L_n, \phi(z)] = D_n\phi = (z^{n+1}\partial_z + h(n+1)z^n)\phi_h(z) \tag{3.1}$$

realises a representation of the centralless Witt subalgebra -"evaluated" rep, trivial center - analog of the "left action" generators.

- The $sl(2)$ conformal subalgebra is realised as

$$D_{-1} = \partial_z \,,\ D_0 = z\partial_z + h \,,\ D_1 = z^2\partial_z + 2hz$$

Invariance of correlators under this finite subalgebra -Ward identities

$$\sum_{a=1}^{n} D_n^{(a)} G_n(z_n, \ldots, z_1) = 0 \,, \quad L_n|0\rangle = 0 = \langle 0|L_{-n} \quad \text{for } n = 0, \pm 1 \tag{3.2}$$

determines the 2- and 3-point correlators up to overall constants

$$\begin{aligned} &G_1 = \langle 0|\phi_h(z)|0\rangle = \delta_{h,0} \,, \\ &G_2 = \langle 0|\phi_{h_1}(z_2)\phi_{h_2}(z_1)|0\rangle = \delta_{h_1,h_2}\, C_{h_1h_1}\,(z_{21})^{-2h_1} \,, \\ &G_3 = \ \langle 0|\phi_{h_3}(z_3)\phi_{h_2}(z_2)\phi_{h_1}(z_1)|0\rangle = C_{h_3h_2h_1}\ z_{21}^{-h_{12}^3} z_{32}^{-h_{23}^1} z_{13}^{-h_{31}^2} \end{aligned} \tag{3.3}$$

3.2. Fusion rules from decoupling of singular vectors

The factorisation of the singular vectors leads to additional constraints on $C_{h_3h_2h_1}$

Take

$$\langle h_3|\phi_{h_2}(z)|h_1\rangle = \lim_{z_3\to\infty} z_3^{2h_3} G_3(z_3, z, 0) = C_{h_3h_2h_1} z^{-h_{12}^3}$$

for the particular value $h_1 = 3t/4 - 1/2$ and using that $\langle h|L_{-n} = 0$ for $n \geq 1$, impose

$$0 = \langle h_3|\phi_{h_2}(z)(L_{-1}^2 - t\,L_{-2})|h_1\rangle \Rightarrow$$

$$\begin{aligned}
&0 = (D_{-1}^2 + b^2 D_{-2})\langle h_3|\phi_{h_2}(z)|h_1\rangle = (\partial_z^2 + \frac{b^2}{z^2}(z\partial_z - h_2))\, z^{-h_{12}^3} \\
&\Rightarrow a(a-1) + t(a - h_2) = 0\,,\ a = -h_{12}^3 \\
&\Rightarrow\ a = \alpha_2 b = J_2 t\,,\ b(\alpha_0 - \alpha_2) = (\tfrac{1}{t} - 1 - J_2)t\ \Rightarrow \\
&h_{J_3} = \alpha_2(\alpha_2 - \alpha_0) + \tfrac{b}{2}(\tfrac{b}{2} - \alpha_0) + a = h(\alpha_2 \pm \tfrac{b}{2}) = h_{J_2 \pm \frac{1}{2}} \\
&\Rightarrow\ C_{h_{J'} h_J h_{J=1/2}} = 0 \text{ unless } J' = J \pm 1/2
\end{aligned} \tag{3.4}$$

If $J_2 = 0$ - use level 1 singular vector - additional constraint, so that only one ($J' = J+1/2$) of the two terms above survives.

Thus (3.4) is reminiscent of the rule of decomposition of the tensor product of $sl(2)$ irreps

Ex. : In the finite $sl(2)$ problem for the invariant 3-point function the analog of (3.4) the raising generator $(E^-)^{2j+1}$ is represented as a differential operator ∂_x^{2j+1} and the decoupling condition on the 3-point function $\langle j_3|\phi_{j_2}(x)|j_1\rangle = x^{j_{12}^3}$ reads (choosing $j_1 \leq j_2$)

$$a = j_{12}^3\,,\quad a(a-1)....(a-2j_1) = 0\ \Rightarrow j_3 = j_1 + j_2\,,\ j_1 + j_2 - 1\,,\ \ldots j_2 - j_1 \tag{3.5}$$

tensor product decomposition rule - 3-point function - Clebsch-Gordan coeffs in a functional realisation of the irreps of $sl(2)$, in which fields - polynomials of a complex variable.

In particular the decoupling of singular vectors of the module $V(h_j, c)$ for $2j$ - positive integer and arbitrary generic $c \neq \mathbb{Q}$ leads to the standard $sl(2)$ tensor product decomposition rule. More generally, for the degenerate representations $V(h_{2j+1,2j'+1}, c)$ - the rule is that of $sl(2) \times sl(2)$. Then (3.5) is recovered for $b \to 0$.

• We arrive at an important notion - the "fusion rule". What happens for rational values of $t = b^2$, the minimal series, which consists of a finite set of representations.? The second singular vector - additional restriction on admissible representations - truncated rule , e.g., for the subset $\{h_{2j+1,1}\}$ satisfying (2.8) the rule reads

$$L_{h_{j_1}} \star L_{h_{j_2}} = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2, p-2-j_1-j_2)} L_{hj} = \sum_j N_{j_1 j_2}^j\, L_{hj} \tag{3.6}$$

NB: $\mathbb{Z}_2$ symmetry - truncates automatically the classical rule.

The *fusion rule multiplicities* $N_{j_1 j_2}^{j}$ (in this particular case $N_{j_1 j_2}^{j} = 0, 1$) can be considered as structure constants of a finite commutative matrix algebra - "fusion algebra".

Example: $b^2 = 3/4 \to c = \frac{1}{2}$ - (critical) Ising model.

$h_{11} = h_{32} = 0\,,\ h_{21} = h_{22} = \frac{1}{16}\,,\ h_{31} = h_{12} = \frac{1}{2},$
corresponding fields $\mathrm{I}\,,\ \sigma, \epsilon$, fusion rules - use the intersection of $\mathbb{Z}_2$ related

$$\sigma \star \sigma = \mathrm{I}\ + \epsilon\,, \quad \sigma \star \epsilon = \sigma\,, \quad \epsilon \star \epsilon = \mathrm{I}$$

fusion matrices

$$N_1 = \mathrm{I}_3\,,\ N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,,\ N_3 = N_2^2 - \mathrm{I}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \tag{3.7}$$

$$N_2^3 - 2N_2 = 0\,, \quad N_3^2 - N_1 = 0$$

We can associate with this set of matrices a graph with adjacency matrix N_2

$$\bullet - - - \bullet - - - \bullet$$

This "fusion" graph which encodes the fusion rules of the model, coincides with a root graph - the Dynkin diagram of the finite dimensional algebra A_2.
We furthermore observe that the eigenvalues of $N_2\,, N_3$ are $\{0, \pm\sqrt{2}\}$ and $\{-1, 1, 1\}$ respectively and that these matrices are diagonalised

$$N_{ij}^{k} = \sum_p \frac{S_{ip}}{S_{1p}}\, S_{jp} S_{kp}^{*}$$

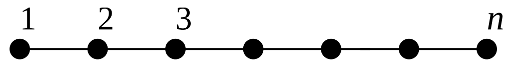
by a symmetric unitary matrix S. The ratios $\chi_j(p) = S_{jp}/S_{1p}$ serve as 1-dim representations (characters) of the fusion algebra

$$\chi_i\, \chi_j = \sum_k N_{ij}^{k}\, \chi_k$$

Ex: Show that these characters can be identified with the $SU(2)$ characters (1.13) choosing a rational angle $\phi = \frac{3r}{4}\pi$ and that the matrix S is given by

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}\,. \tag{3.8}$$

In general the fusion graph looks like the root graph (Dynkin diagram) of $sl(n+1)$.



3.3. BPZ differential equations for the 4-point functions

- Primary fields - global action $z \to z' = z'(z)$

$$\phi'_h(z') = \big(\frac{dz'}{dz}\big)^{-h} \phi_h(z) \tag{3.9}$$

In particular -projective transformation $SL(2)/\mathbb{Z}_2$

$$\begin{aligned} z' &= \frac{az+b}{cz+d}\,,\ ad-bc=1\,,\\ \phi'_h(z') &= (cz+d)^{2h}\,\phi_h(z) \end{aligned} \tag{3.10}$$

If (3.10) holds but (3.9) is in general modified - quasiprimary fields.

Choose a transformation (a,b,c,d) s.t.

$$z_a \to z'_a = \frac{z_{a1} z_{n3}}{z_{31} z_{na}}\,, a = 1,2,3,\dots n\,,\ \text{i.e., } (0, z'_2, 1, z'_4, \dots, z'_{n-1}, \infty).$$

For the 4-point matrix element denoting $z'_2 = z$ we get

Details:

$$b = -az_1,\ d = -cz_4\,,\ \frac{a}{c}\frac{z_{21}}{z_{24}} = z\,, \frac{a}{c}\frac{z_{31}}{z_{34}} = 1\,,\quad z = \frac{z_{21}z_{43}}{z_{24}z_{42}}$$

$$\text{from } ad-bc=1 \to\ ac = \frac{1}{z_{41}}\,,$$

$$\Rightarrow c^2 = \frac{z_{31}}{z_{41}z_{43}}\,, az_4+b = c\frac{z_{34}z_{41}}{z_{31}}$$

$$\begin{aligned} G_4(\{z_i\}) &= \lim_{z'_4\to\infty} (z'_4)^{2h_4}(cz_4+d)^{-2h_4}\prod_{i=1}^{3}(cz_i+d)^{-2h_i}\ G_4(z')\\ &= (az_4+b)^{2h_4}\prod_{i=1}^{3}(c^2 z_{i4}^2)^{-h_i}\ W_4(z') = (c^2)^{h_4-\sum_1^3 h_i}(z_{41})^{2h_4}\prod_{i=1}^{3}(z_{i4})^{-2h_i}\ f(z)\\ &= \frac{z_{13}^{h_4-\sum_{a=1}^{3}h_a}\ z_{14}^{h_2+h_3-h_1-h_4}\ z_{34}^{h_1+h_2-h_3-h_4}}{z_{24}^{2h_2}}\ f(z) \end{aligned} \tag{3.11}$$

The relation (3.11) implies that if we set $z_1 = 0$ but keep unfixed z_3, the limit $z_4^{2h_4}$ of the l.h.s. can be represented as

$$W_4 = \langle h_4|\phi_{h_3}(z_3)\phi_{h_2}(z_2)|h_1\rangle = z_3^a f(z)\,,\ a = h_4 - h_1 - h_2 - h_3 \tag{3.12}$$

where $f(z)$ is an arbitrary function of $z = \frac{z_2}{z_3}$.

Further restrictions arise if at least one of the fields corresponds to a degenerate representation. For instance, choosing $h_1 = h_{2,1} = 3b^2/4 - 1/2$, the decoupling of the 2 level singular vector in the Verma module $V(h_1, c)$, as in (3.4), leads to a 2-nd order ordinary linear differential eqn for (3.12):

$$\begin{aligned}
&\big((\partial_3 + \partial_2)^2 + t(\frac{\partial_3}{z_3} + \frac{\partial_2}{z_2} - \frac{h_3}{z_3^2} - \frac{h_2}{z_2}\big) z_3^a f(z) = 0 \quad \Rightarrow \\
&\big((1-z)^2\partial_z^2 + (1-z)(2a - 2 + t + tz)\partial_z + a(a-1+t) - h_3 - \frac{th_2}{z^2}\big) f(z) = 0\,,
\end{aligned} \tag{3.13}$$

This eqn can be transformed to the hypergeometric linear differential equation

$$\big(z(1-z)\partial_z^2 + (\gamma - z(1+\alpha+\beta))\partial_z - \alpha\,\beta\big) g(z) = 0$$

substituting

$$\begin{aligned}
&f(z) = z^u (1-z)^v g(z)\,,\ \Rightarrow \\
&\quad u = 2\alpha_1\alpha_2\,,\quad v = 2\alpha_2\alpha_3 - (\alpha_{1234} - 1/b)\alpha_{123}^4 = a + b^2/2 - b\alpha_4\,, \\
&\quad \gamma = 2b(\alpha_2 + \alpha_1)\,,\quad \alpha = b\,\alpha_{123}^4 = 2b\alpha_{123} - 1 - (b\alpha_{1234} - 1)\,, \\
&\quad\ \beta = 1 - b\,\alpha_{134}^2 = 2b\alpha_2 - (b\alpha_{1234} - 1)\,.
\end{aligned} \tag{3.14}$$

The equation has generically two linearly independent solutions $g_1\,, g_2$

$$\begin{aligned}
&g_1 = {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\,\frac{z^k}{k!}\,,\quad (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \\
&g_2 = z^{1-\gamma}\ {}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z)
\end{aligned} \tag{3.15}$$

Combining (3.12), (3.14), (3.15) and comparing with (3.4) we observe that for small $z = z_2/z_3 \to 0$ the powers of z_2 and z_3 in the leading order correspond to the two terms in the fusion (3.4),
[i.e., for $\alpha_1 = b/2$, $u = 2\alpha_1\alpha_2 = h(\alpha_2 + \alpha_1) - h(\alpha_2) - h(\alpha_1)\,, \alpha_1 = b/2$, $u + 1 - \gamma = b(\alpha_0 - \alpha_2) = h(\alpha_2 - \alpha_1) - h(\alpha_2) - h(\alpha_1)$]

$$W_4 = \sum_{\pm} \langle h_4 | \phi_{h_3} | h_\pm \rangle \; \langle h_\pm | h_2 | h_1 \rangle + ...$$
$$h_\pm = h(\alpha_2 \pm b/2) \tag{3.16}$$

The two solutions for (3.12) with the small distance behaviour (3.16) determine two "conformal blocks". In operator form, we get the operator product expansion (OPE) of primary fields

$$\begin{aligned}\phi_{h(\alpha_2)}(z)\, \phi_{h(b/2)}(0)|0\rangle &= \sum_{\pm} \frac{C_{h_2 h_1}{}^{h_\pm}}{z^{h(\alpha_2)+h(b/2)-h(\alpha_2 \pm b/2)}} \phi_{h(\alpha_2 \pm b/2)}(z)|0\rangle + ...\\ &= \sum_{\pm} C_{h_2 h_1}{}^{h_\pm} \langle h_\pm | \phi(z_2) | h_1 \rangle \phi_{h(\alpha_2 \pm b/2)}(z)|0\rangle + ...\end{aligned} \tag{3.17}$$

with OPE coefficients $C_{h_2\, h_1\, h_\pm} = C_{h_2\, h_1}{}^{h_\pm}\, C_{h_\pm h_\pm}$.

Example:

In the Ising model $b^2 = 3/4$ with all 4 fields equal to σ, i.e., $\alpha_i = b/2$, we get, using the general formula

$${}_2F_1(a, a-\frac{1}{2}; 2a; z) = \Big(\frac{1+\sqrt{1-z}}{2}\Big)^{1-2a}$$

that the two solutions read

$$\begin{aligned} g_1 &= {}_2F_1(\frac{3}{4}, \frac{1}{4}; \frac{3}{2}; z) = 2\Big(\frac{1-\sqrt{1-z}}{2z}\Big)^{\frac{1}{2}},\\ g_2 &= z^{-\frac{1}{2}}\, {}_2F_1(\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; z) = \Big(\frac{1+\sqrt{1-z}}{2z}\Big)^{\frac{1}{2}},\end{aligned} \tag{3.18}$$

Summary:

- "Degenerate" representations of Vir - obtained as irreducible factors of degenerate Verma modules are analogous to finite irreps of finite dimensional algebras - factorisation leads to equations restricting the correlators including such fields.

Literature: [1].

4. Symmetry as OPE, field descendants

Up to here - we had concentrated on the primary fields. Let see what consequences follow for the descendants

$$T(z) = \sum_{n\in\mathbb{Z}} L_{-n} z^{n-2} \Longleftrightarrow L_n = \frac{1}{2\pi i}\oint_{C_0} T(z) z^{n+1}\,; \quad T(0)|0\rangle = L_{-2}|0\rangle \tag{4.1}$$

the power of z is fixed by the commutator with L_{-1}. Computing the 2-point function,

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{2}\frac{1}{z_{12}^4}$$

The field $T(z)$ generating the Virasoro algebra is the chiral component of the symmetric, conserved and traceless tensor of dimension 2- the energy-momentum (stress) tensor $T_{\mu\nu}$ in 2-dimensional euclidean space-time. It has two independent components,

$T(z) = T_{zz} = \frac{1}{2}T^{\bar{z}\bar{z}} = T_{12} - iT_{12}$, and $\overline{T}(\bar{z}) = T_{\bar{z}\bar{z}} = \frac{1}{2}T^{zz} = T_{12} + iT_{12}$,

satisfying

$$\partial_{\bar{z}} T(z) = 0 = \partial_z \overline{T}(\bar{z})$$

Tracelessness ensures that (classically) the theory is conformal invariant since the currents $j_\mu = T_{\mu\nu}\epsilon^\nu$ corresponding to the conformal coordinate transformations are conserved, $\partial\cdot j = T_{\mu\nu}\partial^\mu\epsilon^\nu = T^\mu_\mu \partial\cdot\epsilon$.

The field $T(z)$ is a quasiprimary field of dimension 2 (i.e., it is a representation of the finite dimensional conformal group), but it is not primary. Indeed the would be vacuum state $T(0)|0\rangle = L_{-2}|0\rangle$ is not annihilated by the positive mode L_2 unless $c = 0$, thus T is a descendant of the identity field; the family of the identity primary field can be identified with the enveloping algebra of Virasoro (modulo the submodule generated by the level level 1, since $L_{-1}|0\rangle = 0$ in the irreducible module, etc.).

We can compute the modified analog of (3.1),

$$[L_n, T(z)] = \big(z^{n+1}\partial_z + 2(n+1)z^n\big)T(z) + \frac{c}{12}\partial_z^3 z^{n+1} \tag{4.2}$$

The global transformation of the quasiprimary field T reads

$$T'(w) = \Big(\frac{dw}{dz}\Big)^{-2}(T(z) - \frac{c}{12}\{w;z\}) = \Big(\frac{dw}{dz}\Big)^{-2} T(z) + \frac{c}{12}\{z;w\} \tag{4.3}$$

where

$$\{w;z\} = \frac{w'''}{w'} - \frac{3}{2}\Big(\frac{w''}{w'}\Big)^2, \quad w' = \frac{dw}{dz}$$

is the Schwarzian derivative. It vanishes identically for the subset of projective transformations (3.10).

Ex: Check that the group property of the transformation (4.3) requires the relations satisfied by the Schwarzian derivative

$$\Big(\frac{dw_2}{dz}\Big)^2\{w_1;w_2\} = \{w_1;z\} - \{w_2;z\}$$

i..e, the representation of composition of maps $z \to w_1 \to w_2 \to z$ same as $z \to w_2$ should be a composition of representations.

$$\begin{aligned} U_{w_2}\phi(z)U_{w_2}^{-1} &= (\frac{\partial w_2}{\partial z})^h\phi(w_2) - \frac{c}{12}\{w_2;z\} \\ &= U_{w_2(w_1)}\phi(z)U^{-1}_{w_2(w_1)} = U_{w_2}U_{w_1}\phi(z)U_{w_1}^{-1}U_{w_2}^{-1} \\ &= U_{w_2}((\frac{\partial w_1}{\partial z})^h\phi(w_1) - \frac{c}{12}\{w_1;z\})U_{w_2}^{-1} \end{aligned} \tag{4.4}$$

etc. For $w_2 = z$ we get the inverse relation used in the last equality in (4.3).

Ex: Compute (4.3) for a transformation of the plane to a cylinder of circumference L where $w = \frac{L}{2\pi}\log z$, $z = e^{\frac{2\pi}{L}w}$, $w \sim w + ikL$

$$\begin{aligned} &\frac{dz}{dw} = \frac{2\pi}{L}z\,,\ \{z;w\} = -\frac{1}{2}(\frac{2\pi}{L})^2 \\ &T^{\rm cyl}(w) = \big(\frac{2\pi}{L}\big)^2\Big(T^{\rm plane}(z)\,z^2 - \frac{c}{24}\Big) \\ &\Rightarrow \frac{2\pi}{L}(L_0^{\rm plane} - \frac{c}{24}) = \frac{2\pi}{L}(\oint T(z)z - \frac{c}{24}) = \frac{1}{2\pi i}\oint dw T^{(cyl)}(w) = H^{(cyl)} \end{aligned} \tag{4.5}$$

a circle in the z-plane $z \to e^{2\pi i}z$ goes to $w \to w + iL$; ; if w=$t + i\sigma$, $0 \le \sigma \le L$,

$H^{(cyl)} + \bar{H}^{(cyl)}$ - generator of "time" translations; $\partial_t = z\partial_z + \bar{z}\partial_{\bar{z}}$, Hamiltonian.

4.1. OPE expansion of T and a field

In general we can express the $n+1$-function with one field T inserted in terms of the n-point function. Let us split the tensor T into two parts annihilating the ket or bra vacuum state respectively

$$\begin{aligned} T(w) = T^-(w) + T^+(w) &= \sum_{n\leq -2} L_n w^{-n-2} + \sum_{n>-2} L_n w^{-n-2}\,, \\ T^+(w)|0\rangle = 0 &= \langle 0|T^-(w) \end{aligned} \tag{4.6}$$

There is some arbitrariness in this choice since the three projective generators $L_n\,, n = 0, \pm 1$ annihilate both vacuum states. Then we compute using (3.1),

$$T(w)\phi(z) = T^-(w)\phi_h(z) + \phi_h(z)T^+(w) + [T^+(w), \phi_h(z)]\,,$$

$$[T^+(w), \phi_h(z)] = \frac{1}{w}\sum_{n=0}\Big(\big(\frac{z}{w}\big)^n\partial_z + h\,\big(\partial_z\big(\frac{z}{w}\big)^n\big)\Big)\phi_h(z) = \frac{h\,\phi_h(z)}{(w-z)^2} + \frac{\partial_z\phi_h(z)}{w-z}\,,\ |w|>|z|\,,\Rightarrow$$

$$T(w)\phi(z) = \frac{h\,\phi_h(z)}{(w-z)^2} + \frac{\partial_z\phi_h(z)}{w-z} + T^-(w)\phi_h(z) + \phi_h(z)T^+(w)\,,\ |w|>|z| \tag{4.7}$$

Repeating (4.7) n times with $|z_0| > |z_i|$ we obtain

$$\begin{aligned} &T(z_0)\phi_{h_1}(z_1)\dots\phi_{h_n}(z_n) \\ &= \sum_i\Big(\frac{1}{z_{0i}}\partial_i + \frac{h_i}{z_{0i}^2}\Big)\phi_{h_1}(z_1)\dots\phi_{h_n}(z_n) + :T(z_0)\phi_{h_1}(z_1)\dots\phi_{h_n}(z_n): \\ &\text{or, for the "time ordered correlator } |z_1|>|z_2|>...|z_n|\,, \\ &\langle T(z_0)\phi_{h_1}(z_1)\dots\phi_{h_n}(z_n)\rangle = \sum_{n\geq -1} z_0^{-n-2}\sum_i D_{n;i}\langle\phi_{h_1}(z_1)\dots\phi_{h_n}(z_n)\rangle \\ &= \sum_i\Big(\frac{1}{z_{0i}}\partial_i + \frac{h_i}{z_{0i}^2}\Big)\langle\phi_{h_1}(z_1)\dots\phi_{h_n}(z_n)\rangle \end{aligned} \tag{4.8}$$

Note that if we had made another choice of the splitting in (4.6), say, by including the three $sl(2)$ generators in the T^- part, would had led to the additional subtraction of the corresponding Ward identities annihilating the n-point correlator and hence to the same result in the r.h.s. of (4.8).

What happens for a different relative order, e.g., for $|z_0| < |z_i|$ for any i. Moving T to the left we have to use

$$\begin{aligned} \phi(z)T(w) &= T^-(w)\phi_h(z) + \phi_h(z)T^+(w) + [\phi_h(z), T^-(w)] \\ &= T^-(w)\phi_h(z) + \phi_h(z)T^+(w) + \frac{h\,\phi_h(z)}{(w-z)^2} - \frac{\partial_z\phi_h(z)}{z-w}\,,\ |z|>|w|; \end{aligned} \tag{4.9}$$

Clearly the r.h.s. if compared with (4.7) does not depend on the radial ordering. Because of the same reason the result does not change if $T(z)$ is put at any arbitrary point between the fields. Similarly, using (4.2) along with (4.7), we can reduce any commutator with insertion of several fileds T to the n-point correlator .

NB: However for the opposite OPE

$$\begin{aligned}\phi(w)T(z) &= T^-(z)\phi_h(w) + \phi_h(w)T^+(z) + [\phi_h(w), T^-(z)] \\ &= T^-(z)\phi_h(w) + \phi_h(w)T^+(z) + \frac{h\,\phi_h(w)}{(w-z)^2} - \frac{\partial_w\phi_h(w)}{w-z}\,,\ |w|>|z|; \\ &= T^-(z)\phi_h(w) + \phi_h(w)T^+(z) + (h/2-1)\partial^2\phi(z) + ... \\ &+ \frac{h\,\phi_h(z)}{(w-z)^2} + \frac{(h-1)\partial_z\phi_h(z)}{w-z}\end{aligned} \tag{4.10}$$

• The relation (4.7) represents the expansion in powers of $(w-z)$ of the (radial) product of $T(w)$ with a primary field $\phi_h(z)$, the coefficients of which introduce new operators, to be denoted $(T\phi_h)^{(l)}(z)$

$$\begin{aligned}T(w)\phi_h(z) &= \sum_{l=-\infty}^{2} \frac{(T\phi_h)^{(l)}(z)}{(w-z)^l} \\ &= \frac{h\,\phi_h(z)}{(w-z)^2} + \frac{\partial_z\phi_h(z)}{w-z} + \sum_{l=0}(T\phi_h)^{(-l)}(z)(w-z)^l \\ &\sim \frac{h\,\phi_h(z)}{(w-z)^2} + \frac{\partial_z\phi_h(z)}{w-z}\end{aligned} \tag{4.11}$$

The terms omitted in the last line are regular for coinciding $w \to z$ and do not contribute to the correlator as in (4.8).

Vice versa, we can recover the commutator (3.1) by contour integration inserting the OPE (4.11)

$$\oint_{C_z} \frac{dw}{w} w^{n+2} R(T(w)\phi_h(z)) = \oint_{C_z} dw\, w^{n+1} \sum_{p=1} \frac{(T\phi_h)^{(p)}(z)}{(w-z)^p} = \sum_{p\geq 1} \big(\frac{\partial_z^{p-1}}{(p-1)!} z^{n+1}\big)\,(T\phi_h)^{(p)}(z)$$

Only the singular part of the OPE contributes to the r.h.s. and we recover the two terms in the r.h.s of (3.1). On the other hand, the contour integral over the radially ordered product in the l.h.s. can be rewritten as the difference of integrals over two contours with reversed order of the operators

$$\oint_{C_{0,|w|>|z|}} \frac{dw}{w} w^{n+2} T(w)\phi_h(z) - \oint_{C_{0,|w|<|z|}} \frac{dw}{w} w^{n+2} \phi_h(z)T(w) = [L_n, \phi_h(z)] \tag{4.12}$$

where the second contour encloses only the point 0, while the first contour encloses both 0 and z; it can be deformed to a contour around the point of infinity on the the Riemann sphere. The same argument is applied to an n-point function with a contour now enclosing all the arguments z_i. Then for the projective generators $L_{\pm 1,0}$, which annihilate the vacua, the l.h.s. vanishes and the r.h.s. recovers the three Ward identities (3.2).

- The first of the finite coefficients in the OPE , i.e., the constant term, defines the *normal product*

$$(T\phi_h)(z) = (T\phi_h)^{(0)}(z) =: T\phi_h : (z)$$

i.e., the leading term in the product at coinciding coordinates, after subtraction of the singular terms in the OPE, according to (4.7). Vice versa, similarly

$$\begin{aligned}(T\phi_h)(z) &= \lim_{w\to z} \big(T(w)\phi_h(z) - \text{singular terms}\big) \\ &= \frac{1}{2\pi i}\oint_{C_z} dw\, \frac{R(T(w)\phi_h(z))}{w-z} \\ &= \frac{1}{2\pi i}\oint_{C_{z,0}} dw \sum_{k=0} (\frac{z}{w})^k \frac{T(w)\phi_h(z)}{w} - \frac{1}{2\pi i}\oint_{C_0} dw \sum_{k=0} (\frac{w}{z})^k \frac{\phi_h(z)T(w)}{-z} \\ &= \sum_{n\geq 2} L_{-n} z^{n-2}\phi_h(z) + \phi_h(z) \sum_{n>-2} L_n z^{-n-2}\end{aligned} \tag{4.13}$$

This is a quasiprimary operator of dimension $h+2$, a descendant of level 2 of the primary field ϕ_h, in agreement with

$$(T\phi_h)(0)|0\rangle = L_{-2}|h\rangle$$

Note that from (4.10) we have

$$(\phi T)(z) = (T\phi)(z) + (h/2-1)\partial^2\phi$$

i.e., the normal product is not commutative.

Furthermore

$$(T\phi_h)^{(-k)}(z) = \phi_h^{(-k-2)}(z) = \frac{1}{k!}(\partial_z^k T^-(z)\phi_h(z) + \phi_h(z)\partial_z^k T^+(z)) = \frac{1}{k!}(\partial_z^k T\phi_h)(z)$$

(the derivative acts on T only). They are descendants of dimension $h+2+k$

$$\phi_h^{(-k-2)}(0)|0\rangle = L_{-k-2}|h\rangle\,, k\geq 0$$

Thus the OPE expansion (4.11) defines a subset of the descendants of the primary ϕ_h, sometimes denoted $(L_{-n}\phi)(z)$

$$\phi_h^{(-1)}(z) = (T\phi_h)^{(1)}(z) = \partial_z \phi_h(z)\,,\ \phi_h^{(-2)}(z) = (T\phi_h)(z)\,,\ \ldots,\ \phi_h^{(-2-k)}(z) = \frac{1}{k!}(\partial^k T\phi_h)(z)$$

and $(T\phi))^{(2)}(z) = h\phi(z)$, with corresponding states $L_{-1}|h\rangle, \ldots L_{-n}|h\rangle \ldots$. In particular we can use the form of the level 2 singular vector

$$\partial_z^2 \phi_{h_{2,1}}(z) - t(T\phi_{h_{2,1}})(z)$$

to derive the factorisation condition inserting the singular vector at an arbitrary point.

- The infinitesimal transformations of $\phi^{(-k)}(z)$ includes besides the standard two terms in the commutator of the primary field - with h replaced by $h+k$, also additional terms involving the fields $\phi^{(-l)}(z)$ with $0 \leq l \leq k$. E.g. from (4.13) we compute

$$[L_m, \phi_h^{(-k)}(z)] = \Big(z^{m+1}\partial_z + (h+k)\big(\partial_z z^{m+1}\big)\Big)\phi^{(-k)}(z) +$$

$$(\frac{c(k^3-k)}{12} + 2kh)\frac{1}{(k+1)!}\partial_z^{k+1} z^{m+1}\phi_h(z) + \sum_{l=1}^{k-1}\frac{k+l}{(l+1)!}\partial^{l+1} z^{m+1}\phi_h^{(l-k)}(z)$$

which in particular reproduces (4.2) for $h = 0$.

Using these modified transformation laws as a starting point, we can compute the OPE $T(w)\phi_h^{(-k)}(z)$, yielding new descendants $\phi_h^{(-p,-k)}(z)$, and thus build recursively any state in the Verma module over $\phi(z)$. The highest singularity will be of order $2+k$, in front of the primary filed ϕ_h contribution. This OPE demonstrates the action of the algebra on the field module

$$T(w)\phi_h(z) = \sum_{m=-\infty}^{\infty}(w-z)^{m-2}\mathcal{L}_{-m}(z)\phi_h(z) \tag{4.14}$$

where

$$(L_{-l}\phi)(z) = \mathcal{L}_{-l}(z)\phi_h(z) := (T\phi_h)^{(-l+2)} = \frac{1}{2\pi i}\int_{C_z}\frac{T(w)\phi(z)}{(w-z)^{l-1}}$$

or

$$\mathcal{L}_n(z) = e^{zL_{-1}}\, L_n\, e^{-zL_{-1}} \tag{4.15}$$

with the properties for a primary $\phi_h(z)$

$$\begin{aligned} &\mathcal{L}_n(z)\phi_h(z) = 0 \text{ for } n > 0 \\ &\mathcal{L}_0(z)\phi_h(z) = h\phi_h(z)\,, \mathcal{L}_{-1}(z)\phi_h(z) = \partial_z\phi_h(z) \end{aligned} \tag{4.16}$$

so that that the sum in (4.14) terminates up to $m = -2$. Similarly from the explicit computation of the deformed commutator of $\phi_h^{(-k)}(z)$ and then the new OPE we get

$$T(w)\phi_h^{(-k)}(z) = T(w)\mathcal{L}_{-k}(z)\phi_h(z) = \sum_{m=-k}^{\infty} (w-z)^{m-2}\mathcal{L}_{-m}(z)\mathcal{L}_{-k}(z)\phi_h(z) \qquad (4.17)$$

and in particular for $1 \leq l \leq k$

$$\begin{aligned} \mathcal{L}_l(z)\mathcal{L}_{-k}(z)\phi_h(z) &= \frac{1}{2\pi i}\oint_{C_z} (w-z)^{l+1} T(w)(L_{-k}\phi_h)(z) \\ &= (l+k)\mathcal{L}_{-(k-l)}(z)\phi_h(z) + \delta_{l,k}\frac{c}{12}(k^3-k)\phi_h(z)) \end{aligned} \qquad (4.18)$$

while the action of $\mathcal{L}_n$ on a descendant of level k gives zero for $n > k$. E.g., for $h = 0$ using the commutator (4.2), or directly (4.17) for $\phi_{h=0}^{(-2)}(z) = (TI)(z) = T(z)$ we obtain

$$\begin{aligned} T(w)T(z) &= \frac{c/2}{(w-z)^4} + \frac{2\,T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \text{regular} \\ &= \big(\frac{c}{12}\partial_z^3 + 2T(z)\partial_z + \partial_z T(z)\big)\frac{1}{w-z} + \text{regular} \end{aligned} \qquad (4.19)$$

Here the normal product is an operator of dimension 4, a descendant of level 4 of the identity operator

$$\begin{aligned} &: TT :_m = \sum_{n\geq 2} L_{-n}L_{m+n} + \sum_{n\geq -1} L_{m-n}L_n\,, \\ &\tfrac{1}{k!}(\partial^k TT)(0)|0\rangle = \tfrac{1}{k!}\sum_{p=0}^{k}\binom{k}{p}(-1)^p L_{-1}^p L_{-2} L_{-1}^{k-p} L_{-2}|0\rangle = L_{-k-2}L_{-2}|0\rangle\,,\ k \geq 0\,. \end{aligned} \qquad (4.20)$$

The OPE (4.17) generalises for arbitrary descendent field
$\phi_h^{(-k_1,-k_2,\ldots,-k_n)}(z) = \mathcal{L}_{-k_1}(z)\ldots\mathcal{L}_{-k_n}(z)\phi_h(z)$.

Summary:

1. For any fixed value of the central charge the conformal family $[\phi_h]$ generated by a primary field is in one to one correspondence with the abstract Vir module; the identity family (containing the normal product powers of T), is identified with the universal envelope of the negative mode Virasoro algebra itself.

2.The primary field correlators are sufficient to compute the correlators of the descendants, i.e., the whole information is already provided by the primary fields;

- Comment on locality

  In M_1 locality means that the commutator has support on the line $t_{12} = 0$ - hence it is expressed by derivatives of delta functions; same formulation in euclidean picture using formal delta functions and their derivatives, which replace the powers in the singular part of the OPEs like (4.19). Mutual locality of two formal distributions in the euclidean picture - equivalent to finite number of singular terms in the (radially ordered) OPE; applies to algebra itself and its OPE with the fields. The chiral fields of (half)-integer dimension h provide particular examples of local 2d fields to be discussed later.

4.2. General rules

This OPE example can be generalised whenever the OPE of the two operators is represented by a Laurent series with a finite number of singular terms

Consider a field A of dimension h_A with expansion

$$A(z) = \sum_{m\in\mathbb{Z}} A_m z^{-m-h_A} \quad \text{or} \quad A_m = \oint_{C_0} \frac{dz}{z} A(z)\, z^{m+h_A}$$

and

$$(\dot{A})_n = -(n + h_A) A_n$$

for the modes of the derivative of the field A.

Assign a fermion number

$$F(A) = \begin{cases} 0 & \text{if } A \text{ is bosonic} \\ 1 & \text{if } A \text{ is fermionic} \end{cases}.$$

For two fields A and B define the radially ordered product as

$$\mathcal{R}(A(w)\, B(z)) = \begin{cases} A(w)B(z) & \text{if } |w| > |z| \\ (-1)^{F(A)F(B)} B(z)A(w) & \text{if } |w| < |z| \end{cases}.$$

In fact all operator products considered will be radially ordered and we will drop the symbol $\mathcal{R}$. We assume that the OPE holds

$$A(w)\, B(z) = \sum_{n=-\infty}^{p} \frac{(AB)^{(n)}(z)}{(w-z)^n} \tag{4.21}$$

with the sum running to a finite nonnegative number ($\leq h_A + h_B$) and $(AB)^{(n)}$ are some fields, (composite or descendant). For the normal product we use interchangeably the notation

$$(AB)^{(0)} =: AB := (AB).$$

The singular part of the OPE is called the contraction denoted $\underbrace{A\,B}$, i.e.,

$$A(z)\,B(w) = \underbrace{A(z)\,B(w)} + (AB)(w) + O(z-w).$$

Compare with the definition of the normal product for free, dimension 1 field, for which there is one singular term $(\phi\phi)^{(2)} = 1$ and

$$:\phi\phi:(z) = \lim_{w\to z}(\phi(w)\phi(z) - \langle\phi(w)\phi(z)\rangle) = \lim_{w\to z}(\phi(w)\phi(z) - \frac{1}{(w-z)^2})$$

Similarly to the way we recovered the commutator (4.2) from the OPE (4.11) one derives the commutation relations from the singular part of (4.21), at the last step integrating over z to recover the modes of the second field

$$\begin{aligned} A_n B_m - (-1)^{F(A)F(B)} B_m A_n &= \frac{1}{2\pi i}\oint_{C_0}\frac{dz}{z} z^{m+h_B}\frac{1}{2\pi i}\oint_{C_z}\frac{dw}{w} w^{n+h_A} A(w)B(z) \\ &= \sum_l \frac{1}{2\pi i}\oint_{C_0}\frac{dz}{z} z^{m+h_B}\left(\frac{\partial_z^{l-1}}{(l-1)!} z^{n+h_A-1}\right)(AB)^{(l)}(z) \\ &= \sum_{l=1}\binom{n+h_A-1}{l-1}(AB)^{(l)}_{n+m} \end{aligned}$$

where C_0 is a small circle in the z plane around 0 and C_z is a small circle in the w plane which surrounds z but not 0.

The coefficient functions $(AB)^{(k)}$ in (4.21) are not symmetric in A and B but the following exchange relations hold

$$(-1)^{F(A)F(B)}(BA)^{(n)} = \sum_{m\geq 0}^{\text{finite}} \frac{(-1)^{n+m}}{m!}\partial^m (AB)^{(n+m)} \tag{4.22}$$

There is an analog of the Wick theorem - which is recovered for $k=2$ being the only singular (and constant) term in the OPE); here $k \in \mathbb{Z}_+$

$$(A(BC)^{(0)})^{(k)} = (-1)^{F(A)F(B)}(B(AC)^{(k)})^{(0)} + \sum_{m\geq 0}^{\text{finite}}\binom{k-1}{m}((AB)^{(k-m)}C)^{(m)} \tag{4.23}$$

With these formulae we can compute the OPE expansions of the normal product of fields, if we know the OPEs of the fields themselves.

N.B. One could guess that there is an analogous Wick rule when the composite field is on the left, i.e. the l.h.s. is $((AB)C)^{(k)}$, but this is not true. So for such contractions one has first to use the exchange relation to move the composite field on the right and then to use Wick.

From the definition one immediatelly has another useful identity

$$\partial(AB)^{(k)} = (\dot{A}B)^{(k)} + (A\dot{B})^{(k)}, \qquad (\dot{A}B)^{(k)} = (-k+1)\,(AB)^{(k-1)}$$

NB: There is a Mathematica program [14] which computes OPEs using these rules.

4.3. OPE coefficients of the descendants

The Vir symmetry allows to determine the OPE coefficients of the descendent fields from those of the primary fields. Indeed consider the OPE of two primary fields

$$\phi_{h_1}(z)|h_2\rangle = \sum_{h_3} C_{h_1 h_2}^{h_3} z^{-h_{12}^3}|h_3\rangle\,,\ |h\rangle = \sum_k z^k |h,k\rangle \tag{4.24}$$

the sum over k denotes the contribution of descendants. We apply L_n, $n>0$ to both sides using that $L_n|h_2\rangle = 0$, while on the first field the generator acts by the differential operator (3.1),

$$(D_n\phi_{h_1}(z))\ |h_2\rangle = (z^{n+1}\partial_z + h_1(n+1)z^n)\phi_{h_1}(z)|h_2\rangle = \sum_h C_{h_1 h_2}^{h} z^{-h_{12}^3} L_n|h\rangle\ \Rightarrow$$

Inserting the OPE in the l.h.s., performing the differentiation and equating the coefficients in front of the power $z^{n+k-h_{12}^3}$ in both sides

$$(-h_{12}^3 + h_1(n+1)) + k)|h,k\rangle = L_n|h_3, k+n\rangle\,, n>0 \tag{4.25}$$

The r.h.s. is computed for each descendent state

$\sum \beta^{(n;-n_1,-n_2,\dots -n_k)} L_{-n_1}\dots L_{-n_k}|h_3\rangle\,,\ n_{\geq}\dots \geq n_k.$

Thus we get recursively, for $n=1$, $k=0$

$$L_1\beta^{(1;-1)}L_{-1}|h_3\rangle = (-h_2+h_3+h_1)|h_3\rangle \Rightarrow \beta^{(1;-1)} = \frac{h_{13}^2}{2h_3}$$

Next for $k+n=2$, i.e., $k=1\,,n=1$ and $k=0, n=2$

$$\begin{aligned} &L_1(\beta^{(2;-1,-1)}L_{-1}^2 + \beta^{(2;-2)}L_{-2})|h_3\rangle = (-h_2+h_3+h_1+1)\frac{h_{13}^2}{2h_3}L_{-1}|h_3\rangle \\ &L_2(\beta^{(2;-1,-1)}L_{-1}^2 + \beta^{(2;-2)}L_{-2})|h_3\rangle = (-h_2+h_3+2h_1)|h_3\rangle \end{aligned} \tag{4.26}$$

Performing the commutators in the l.h.s. we get two equations for the unknown coefficients $\beta^{(2;-1,-1)}$ and $\beta^{(2;-2)}$ at level 2, which determine them. This recursive procedure yields any coefficient, although there is no close analytic expression.

4.4. The quasi-primary fields

The OPE expansion can be also rewritten in terms of another basis, that of the *quasi-primary fields* and their derivatives. The quasi-primary fields transform as (3.1) for $n = \pm 1, 0$ and create states $|h\rangle = \phi_h(0)|0\rangle$ which satisfy $L_1|h\rangle = 0$, i.e., they are h.w. states of the finite dimensional subalgebra. The primary fields are apparently also quasi-primary. An example of a quasi-primary which is not primary is provided by the stress tensor $T(z)$ since the anomalous term in (4.2) vanishes for the projective generators; also $L_1 T(0)|0\rangle = L_1 L_{-2}|0\rangle = L_{-1}|0\rangle = 0$. The quasi-primary fields in a given conformal familty and their derivatives provide an equivalent basis.

Assuming that the fields ϕ_{h_1}, ϕ_{h_1} are at least quasi-primary we decompose their product in the basis of the quasi-primaries ϕ_{h_3} and their derivatives,

$$\phi_{h_1}(z)|h_2\rangle = \sum_{h_3,k} \gamma_{h_1 h_2}^{h_3,k} z^{k-h_{12}^3} L_{-1}^k |h_3\rangle \tag{4.27}$$

We can partially solve for the OPE constant reducing it to a constant $\gamma_{h_1 h_2}^{h_3} = \gamma_{h_1 h_2}^{h_3,0}$ depending only on the dimensions of the quasi-primary fields ϕ_{h_3} (which may appear with multiplicity). Indeed apply to both sides L_1 and use that it annihilates the quasiprimary states $|h_2\rangle$ and $|h_3\rangle$. Exploiting in the l.h.s. the action (3.1) on the first field, taking into account the commutator

$$[L_1, L_{-1}^k] = k L_{-1}^{k-1}(2L_0 + k - 1)$$

and comparing the coefficients in both sides we get the recursion relation

$$\begin{aligned} &\gamma_{h_1 h_2}^{h_3,k}(h_3 + h_1 - h_2 + k) = \gamma_{h_1 h_2}^{h_3,k+1}(k+1)(2h_3 + k)\,, \\ &\Rightarrow\ \gamma_{h_1 h_2}^{h_3,k} = \gamma_{h_1 h_2}^{h_3} F_k(h_{31}^2; 2h_3)\,,\ F_k(\alpha;\gamma) = \frac{(\alpha)_k}{k!(\gamma)_k} \end{aligned} \tag{4.28}$$

Thus (4.27) is expressed by the degenerate hypergeometric function ${}_1F_1$

$$\phi_{h_1}(z)|h_2\rangle = \sum_{h_3} \gamma_{h_1 h_2}^{h_3} z^{-h_{12}^3} {}_1F_1(h_3 + h_1 - h_2; 2h_3; zL_{-1})|h_3\rangle \tag{4.29}$$

Literature: [1], [15].

5. Vertex operators, free field (Coulomb gas) construction

Singular vectors and the related differential equations for the correlators of primary fields have a complicated structure in general. Instead - there is an explicit direct construction based on the representation of the tensor T as a composite operator made of more elementary fields.

5.1. Free field realisation, Fock modules

We consider dimension 1 primary field

$$\begin{aligned} J(z_1)J(z_2) &\sim \frac{1}{z_{12}^2} \\ T(z) &= \tfrac{1}{2} : JJ : (z) = - : \partial\phi\partial\phi : (z) \end{aligned} \tag{5.1}$$

This is the chiral component of the energy-momentum tensor of the 2d theory describing a massless free field with action

$$S = \frac{1}{4\pi} \int d^2x (\partial_a \phi)^2$$

and a conserved (classically) current $J^a \sim \partial^a \phi \,, \partial_a J^a = 0$. This is the simplest example of a general construction of T as quadratic normal product of "currents", with values in a Lie algebra, which generate an affine Lie algebra - known as the Sugawara construction; here is the example of an abelian, $\hat{u}(1)$.

The normal product here is the standard one and the Wick theorem is applicable

$$: JJ : (z) = \lim_{\epsilon \to 0} (J(z+\epsilon)J(z) - \frac{1}{\epsilon^2}) = \lim_{\epsilon \to 0} (J(z+\epsilon)J(z) - \langle J(\epsilon)J(0) \rangle) \tag{5.2}$$

and we compute

$$\begin{aligned} \underbrace{T(w)\,J(z)} &= \frac{J(w)}{(w-z)^2} = \frac{J(z)}{(w-z)^2} + \frac{\partial_z J(z)}{w-z} \,, \\ \underbrace{T(w)\,T(z)} &= \frac{: J(w)J(z) :}{(w-z)^2} + \frac{1}{2(w-z)^4} = \frac{1}{2(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T}{w-z} \end{aligned} \tag{5.3}$$

We obtain a Vir algebra of central charge $c = 1$. A straightforward generalisation of such tensor comes from the collection of several free fields, or ϕ^μ is a vector in a D -dimensional space $\mu = 1, 2, ...D$, like in the $D = 26$ bosonic string theory; each of these modes adds 1 to the central charge.

We now deform T by adding a derivative of the current

$$T(z) = \tfrac{1}{2} : JJ : (z) + \frac{\alpha_0}{\sqrt{2}}\, \partial J(z)$$
$$T(w)J(z) \sim -\frac{\sqrt{2}\alpha_0}{(w-z)^3} + \frac{J(z)}{(w-z)^2} + \frac{\partial_z J}{w-z} \tag{5.4}$$

The effect of the added term is to change the central charge to

$$c = 1 \to 1 - 6\alpha_0^2$$

We see that we can identify this parameter α_0 with the parameter discussed before, namely $\alpha_0 = 1/b - b\,, 6\alpha_0^2 = -12 + b^2 - 1/b^2$, or $c = 13 - 6(t + 1/t)\ t = b^2$.

In Fourier modes

$$J(z) = \sum a_n z^{-n-1}\,,$$
with commutation relations (5.5)
$$[a_n, a_m] = n\delta_{n,-m}$$

i.e., a Heisenberg algebra. One defines a Fock space module F_{α,α_0} labelled by α_0 and another parameter α labelling a state annihilated by the positive modes

$$a_n|\alpha\rangle = 0 \text{ for all } n > 0\,,\ a_0|\alpha\rangle = \sqrt{2}\alpha|\alpha\rangle$$
$$F_{\alpha,\alpha_0} = \text{linear span } \{|n_k, \ldots, n_1\rangle = a_{-k}^{n_k} \ldots a_{-1}^{n_1}|\alpha\rangle\}\,, \tag{5.6}$$

The zero mode a_0 has the same eigenvalue on any state of the Fock space since it commutes with all generators.

We can define an action of the Vir algebra in this Fock space, i.e., consider it as a Virasoro module. Namely the generators appearing from the normal product definition are expressed in terms of the Heisenberg generators a_n

$$L_n = \tfrac{1}{2}\big(\sum_{k\geq 1} a_{-k}a_{n+k} + \sum_{k\geq 0} a_{n-k}a_k\big) - (n+1)\frac{\alpha_0}{\sqrt{2}} a_n$$
$$L_0 = \sum_{k\geq 1} a_{-k}a_k + \frac{a_0}{\sqrt{2}}\big(\frac{a_0}{\sqrt{2}} - \alpha_0\big) \tag{5.7}$$

Their commutators with the current modes read

$$[L_n, a_m] = -m\, a_m - n(n+1)\frac{\alpha_0}{\sqrt{2}}\delta_{m+n}$$

or

$$[L_n, J(z)] = \partial_z(z^{n+1}J(z)) - \frac{\alpha_0}{\sqrt{2}}\partial_z^2 z^{n+1}$$

demonstrating the fact that the current J is primary with respect to T only for $\alpha_0 = 0$.

The positive Vir modes annihilate the vacuum state

$$L_n|\alpha\rangle = 0$$

The Fock module is naturally graded by L_0 since

$$[L_0, a_{-m}] = m a_{-m}$$

and the eigenvalues are bounded from below by the eigenvalue of the h.state $|\alpha\rangle$

$$L_0|\alpha\rangle = \frac{a_0}{\sqrt{2}}(\frac{a_0}{\sqrt{2}} - \alpha_0)|\alpha\rangle = \alpha(\alpha - \alpha_0)|\alpha\rangle = h(\alpha)|\alpha\rangle \tag{5.8}$$

We recover the parametrisation of the scaling dimension $h(\alpha)$ in terms of the parameter α, referred to as a "charge" - it is eigenvalue of the zero mode of the current - a conserved charge $a_0 = \frac{1}{2\pi i}\oint dz J(z)$, commuting with the "Hamiltonian" $[L_0, a_0] = 0$.

A dual Fock space F^*_{α,α_0} is defined, build from a dual h.w. state $\langle\alpha|$, normalised so that $\langle\alpha|\alpha\rangle = 1$, and the hermitian conjugated generators a_n^+; conjugation s.t. to ensure $L_n^+ = L_{-n}$,

$$a_n^+ = -a_{-n}\ ,\ n \neq 0\,,\ \ a_0^+ = \sqrt{2}\alpha_0 - a_0\,,$$

we thus identify the dual module F^*_{α,α_0} with the Fock module $F_{\alpha_0-\alpha,\alpha_0}$.

The Fock space character can be computed directly and reproduces for generic values of $\alpha\,, \alpha_0$ the expression we obtained before

$$tr_\alpha q^{L_0 - c/24} = q^{\alpha(\alpha-\alpha_0)-c/24} tr_\alpha q^{\mathbb{N}} = \frac{q^{\alpha(\alpha-\alpha_0)-c/24}}{\phi(q)}$$

$$\langle n|a_{-k}a_k|n\rangle = k n_k\ ,\ \langle n|q^{a_{-k}a_k}|n\rangle = \langle n|q^{k n_k}|n\rangle\ ,\ L_0|n\rangle = (h(\alpha) + \sum_{k=1} k n_k)|n\rangle$$

$$tr_\alpha q^{\mathbb{N}} = tr q^{\sum_{-\infty}^{\infty} a_{-k}a_k} = \sum_{n_1=0}^{\infty} \dots \sum_{n_k} q^{n_1} q^{2n_2} \dots q^{k n_k} \dots = \prod_1^{\infty} \frac{1}{1-q^k}$$

5.2. Vertex operators, screening charges

The 2d euclidean Green function of the free massless field is logarithmic

$$G_2(x_{12}) = \langle\phi(x_1)\phi(x_2)\rangle = -\frac{1}{2}\ln x_{12}^2\,, \quad \Box\, G_2(x) = -2\pi\delta^{(2)}(x)$$

i.e., it is not itself a conformally covariant field of dimension $h = 0$, only its derivative, the current transforms covariantly; for the current corresponding to the chiral part of $\phi(x) = \phi(z) + \phi(\bar{z})$

$$J(z) = \sqrt{2}i\partial_z\phi(z,\bar{z})\,,$$

so that the stress tensor is expressed as

$$T(z) = -:\partial\phi\partial\phi: + i\alpha_0\partial^2\phi$$

The mode expansion of ϕ is an integrated version of that of J; we split it into positive and negative frequency parts

$$\begin{aligned}&\phi(z) = \phi^{(+)}(z) + \phi^{(-)}(z) := \\ &\frac{i}{\sqrt{2}}\big(-a_0\log z + \sum_{n>0}\frac{a_n}{n}z^{-n}\big) + \frac{i}{\sqrt{2}}\big(-iq - \sum_{n>0}\frac{a_{-n}}{n}z^n\big)\,,\ [a_0, q] = -i\end{aligned} \tag{5.9}$$

The exponential $e^{\sqrt{2}i\alpha}$ of the operator q, commuting with all non-zero modes, determine the vacuum state of the Fock module

$$e^{\sqrt{2}i\alpha}|0\rangle = |\alpha\rangle$$

Note the non-diagonal action of the dilatation generator on the massless field

$$\begin{aligned}&[L_0,\phi] = z\partial\phi - \frac{i}{\sqrt{2}}(a_0 - \frac{\alpha_0}{\sqrt{2}})\,, \\ &[L_0, a_0] = 0\end{aligned} \tag{5.10}$$

• Primary fields with respect to T are constructed as vertex operators; here only the chiral part

$$\begin{aligned}V_\alpha(z) =: e^{2i\alpha\phi(z)} := e^{2i\alpha\phi^{(-)}(z)}\ e^{2i\alpha\phi^{(+)}(z)}\,, \\ V_\alpha(0)|0\rangle = e^{\sqrt{2}i\alpha q}|0\rangle = |\alpha\rangle\end{aligned} \tag{5.11}$$

The standard transformation with L_n is checked by direct computation.

Next we compute how the positive and negative parts are exchanged, using for each mode the Heisenberg group relation

$$e^{ca_n} e^{c' a_{-n}} = e^{cc' [a_n, a_{-n}]} \, e^{c' a_{-n}} e^{ca_n}$$

and then summming a logarithmic series $\sum_{n=0} \frac{1}{n} (\frac{z_1}{z_2})^n = -\ln(1 - \frac{z_1}{z_2}) = -\ln z_{21} + \ln z_2$; the last term is compensated by the zero mode commutator, so

$$V_{\alpha_2}^{(+)}(z_2) \, V_{\alpha_1}^{(-)}(z_1) = z_{21}^{2\alpha_1\alpha_2} \, V_{\alpha_1}^{(-)}(z_1) \, V_{\alpha_2}^{(+)}(z_2) \,, \qquad |z_2| > |z_1| \,, \tag{5.12}$$

We can use this in a product of operators. This commutation relation implies the OPE

$$V_{\alpha_2}(z_2) \, V_{\alpha_1}(z_1) \sim z_{21}^{2\alpha_1\alpha_2} \, V_{\alpha_1+\alpha_2}(z_1) = \frac{1}{z_{21}^{h(\alpha_1)+h(\alpha_2)-h(\alpha_1+\alpha_2)}} \, V_{\alpha_1+\alpha_2}(z_1) \,,$$

using that at coinciding points $V_{\alpha_1}^{\pm} V_{\alpha_1}^{\pm} = V_{\alpha_1+\alpha_2}^{\pm}$. We see that the leading contribution in the OPE of the vertex operators is given by a vertex operator of dimension $h(\alpha_3)$ with charge equal to the sum of the charges of the two field in the product, i.e., we get the simple fusion rule

$$\phi_{h_{\alpha_1}} \star \phi_{h_{\alpha_2}} = \phi_{h_{\alpha_1+\alpha_2}}$$

In computing the correlators we have to ensure that the overall phase created by the constant mode of the field q, including the charge $-\alpha_0$ at infinity, is absent $e^{\sqrt{2} i q \sum_i \alpha_i} = 1$ (to reduce to the norm of the vacuum). Thus a necessary condition to have a nonzero correlator is the conservation of the sum of charges

$$\sum_i \alpha_i = \alpha_0 \tag{5.13}$$

the r.h.s. would be zero in the free $c = 1$ case; here $-\alpha_0$ is interpreted as background charge created by a vertex operator $V_{-\alpha_0}$ at infinity; we just omit writing this field explicitly.

For the 2-point function

$$\begin{aligned} &\langle \alpha_0 | V_{\alpha_0-\alpha}(z_1) \, V_\alpha(z_2) | 0 \rangle = z_{12}^{-2h(\alpha)} \langle \, \alpha_0 | \alpha_0 \rangle \,, \qquad |z_1| > |z_2| \,, \\ &\text{and more generally} \\ &\langle \, V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) \, \rangle = \prod_{0 \le i < j \le n} z_{ij}^{2\alpha_i\alpha_j} \,, \; \sum_i \alpha_i = \alpha_0 \end{aligned} \tag{5.14}$$

• The charge conservation condition (5.13) is apparently too restrictive - does not allow e.g. a correlator with 4 fields of the same dimension. To relax this one introduces "screening charges" - dimension 0 fields which change the balance of charges. These are integrated vertex operators $V_{\alpha_\pm}$ of dimension 1 - which commute with Vir - up to boundary terms, which should disappear in correlators, and so preserve the conformal properties. One checks that $h(-b) = 1 = h(1/b)$

$$Q_\pm = \oint_C dz V_{\alpha_\pm}(z)\,,\ \alpha_+ = -b\,, \alpha_- = 1/b\,,\ [L_n, V_{\alpha_\pm}(z)] = \partial(z^{n+1} V_{\alpha_\pm}(z))$$

Then we insert a finite number of screening charges - charge conservation condition (5.13) changed

$$\sum_i \alpha_i = e_0 + mb - \tfrac{n}{b}\,,\quad m, n \in \mathbb{Z}_{\geq 0} \tag{5.15}$$

- Let see on an example how the OPE in the presence of screening charge operator changes

$$\begin{aligned}
&V_\alpha(z_2) \oint_{C_{z_2,z_1}} dw V_+(w) V_\beta(z_1)|0\rangle \\
&= z_{21}^{2\alpha\beta}\,(-1 + e^{-2\pi i 2b\beta}) \int_{z_1}^{z_2} dw (z_2 - w)^{-2b\alpha}(w - z_1)^{-2b\beta} V_{-b}(w) \prod V_{\alpha_i}^{(-)}|0\rangle \\
&= z_{21}^{2\alpha\beta - 2b(\alpha+\beta)+1}\,(-1 + e^{-2\pi i 2b\beta}) \int_0^1 dw (1-w)^{-2b\alpha} w^{-2b\beta} V_+(z_{21}w + z_1) \prod V_{\alpha_i}^{(-)}|0\rangle \\
&\sim \frac{1}{z_{21}^{h(\beta)+h(\alpha)-h(\alpha+\beta-b)}} \frac{(-2i)\, e^{-2\pi i b\beta} \sin \pi 2b\beta\, \Gamma(-2b\beta+1)\Gamma(-2b\alpha+1)}{\Gamma(-2b\alpha - 2b\beta + 2)} V_{\alpha+\beta-b}(z_1)|0\rangle
\end{aligned} \tag{5.16}$$

here the contour C_{z_2,z_1} is chosen from $z_2 + i\epsilon$ above the cut between the two initial points (or above the real axis in the new coordinates), and then back to $z_2 - i\epsilon$ below the cut (below the real axis), surrounding z_1 along an infinitesimally small circle in positive direction; this leads to the prefactor $(-1 + e^{-2\pi i 2b\beta})$. The computation of the integral is reduced to a beta function. Other choices of the contour are possible; this one has the merit that for $\beta = 0$, the coefficient vanishes, in agreement with the fusion of the identity.

We conclude that now the fusion rule has been changed - instead of simply adding charges, now

$$\phi_{h_\alpha} \star \phi_{h_\beta} = \phi_{h_{\alpha+\beta-b}}$$

If one of the fields corresponds to our fundamental degenerate rep, $h = h_{2,1}$, i.e., $\alpha = b/2$, we recover the two fusions coming from the OPE without, or with a screening charge

$$V_{\frac{b}{2}}\,,\ F_{\beta,\alpha_0} \to F_{\beta+\frac{b}{2}}\,,\quad V_{\frac{b}{2}} Q_+\,,\quad F_{\beta,\alpha_0} \to F_{\beta-\frac{b}{2}}$$

the free field representation of the fields recovers automatically the fusion rules we have obtained before from the decoupling of the singular vectors. The two vertex operators act as projections of the field, with a meaning of intertwining Fock spaces.

- More generally we can define an intertwining operator assigning the screening charges to the operator

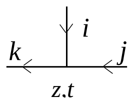
$$V_\alpha^{(m,n)}(z) := V_\alpha(z) \oint_{C_1} dw \dots \oint_{C_{m+n}} du\, V_+(w_1)\dots V_+(w_m) V_-(u_1)\dots V_-(u_n) \qquad (5.17)$$

with contours $C_z^{(+0)}$ from z to z surrounding once the point 0 in positive (anti-clock wise) direction and embedded so that C_{i+1} is inside C_i. We get a sequence of fields of the same dimension - intertwining, chiral vertex operators (CVO)

$$F_{\beta,\alpha_0} \to F_{\beta+\alpha-mb+n/b,\alpha_0} \qquad (5.18)$$

These CVO determine the possible fusion channels of interaction fields, i.e., the possible triples of admissible representations. Here the resulting dimensions are $h(\alpha+\beta-mb+n/b)\,, m,n = 0,1,...;$. This is consistent with our general fusion rule - recall that the charges of the rational (minimal) series are parametrised as $\alpha_{r,s} = \frac{r-1}{2}b - \frac{s-1}{2b}$ with r,s restricted; then actually a finite number since some coefficients are vanishing. To distinguish the interacting vertex operators one indicates the initial and final representation, e.g., O_{ij}^k, graphically

$$\phi_{ij;t}^k(z) \;:\; \mathcal{V}_j \mapsto \mathcal{V}_k \quad t = 1, \cdots, \mathcal{N}_{ij}{}^k \qquad (5.19)$$



These operators are examples of (primary) chiral vertex operators CVO, or intertwining operators, which one associates in general with triples of irreps of the chiral algebra at a given central charge.

In general there is an index $t = 1, 2, \dots \mathcal{N}_{ij}{}^k$ distinguishing different CVO. The non-negative integers $\mathcal{N}_{ij}{}^k$ are interpreted as the multiplicities in the *fusion* $\mathcal{V}_i \star \mathcal{V}_j$ of two representations

$$\mathcal{V}_i \star \mathcal{V}_j = \oplus_k \mathcal{N}_{ij}{}^k \mathcal{V}_k\,. \qquad (5.20)$$

5.3. Cohomological interpretation of the Fock space realisation of the Vir representations

In the rational case - the minimal theories irreducible representations are recovered from the Fock space construction introducing the " multiple screening charges", i.e., the operators $Q_m = V_0^{(m,0)}$ as in (5.17), $1 \le m \le p-1$,

$$
\begin{aligned}
Q_m : \ F_{\alpha_{m,n},\alpha_0} &\rightarrow F_{\alpha_{-m,n},\alpha_0} \,, \\
Q_{p-m} : \ F_{\alpha_{2p-m,n},\alpha_0} &\rightarrow F_{\alpha_{m,n},\alpha_0} \\
Q_m \ Q_{p-m} = 0 & \qquad \text{(BRST property)}
\end{aligned}
\tag{5.21}
$$

this defines a BRST complex

$$
\dots \ F_{\alpha_{2p-m,n}} \stackrel{Q_{p-m}}{\longrightarrow} F_{\alpha_{m,n}} \stackrel{Q_m}{\longrightarrow} F_{\alpha_{-m,n},\alpha_0} \ \dots
$$

where in the middle $\alpha_{m,n}\,, 1 \le m \le p-1\,,\ 1 \le n \le p'-1$. The spectral sequence is exact (Ker =Im) except at the middle Fock space. Thus the irreducible Vir representations are obtained as the non-trivial cohomology

$$
L(h(\alpha_{m,n}), c_{p,p'}) \simeq \text{Ker } Q_m / \text{Im } Q_{p-m}
$$

Alternatively, the same result is obtained from the dual operators $\tilde{Q}_n \,, \tilde{Q}_{2p'-n}$

5.4. 2d formulation

Why "Coulomb gas" : In 2d the electrostatic potential energy of a pair of charges varies logarithmically with the distance between the charges.

Free field lagrangean formulation: Gaussian integral

$$
\begin{aligned}
& e^{-S} = e^{-\frac{1}{4\pi}\int d^2x (\partial_a\phi)^2} = e^{-\frac{1}{2}\int d^2x (\phi - \frac{\Box}{2\pi}\phi)} \,, \ y = \sum_j 2i\alpha_j \delta^{(2)}(x - x_j) \\
& \frac{\int D\phi e^{-\frac{1}{2}\int d^2x(\phi \frac{-\Box}{2\pi}\phi) + \int d^2x \ y(x)\phi(x)}}{\int D\phi e^{-\frac{1}{2}\int d^2x(\phi - \frac{\Box}{2\pi}\phi)}} \\
& = Z e^{\frac{1}{2}y(\frac{-\Box}{2\pi})^{-1}y} = Z \ e^{\sum_{i,j}\alpha_i\alpha_j \log x_{ij}^2} = \prod_{i<j}(x_{ij}^2)^{2\alpha_i\alpha_j} \,,
\end{aligned}
\tag{5.22}
$$

$$
[\langle\phi\phi\rangle = (-\frac{\Box}{2\pi})^{-1} = -\frac{1}{2}\log x^2 \Rightarrow \partial_{\bar{z}}\partial_z(\log z + \log\bar{z}) = \partial_{\bar{z}}\frac{1}{z} = \pi\delta^2(x)]
$$

2d screening charges added as interaction terms

$$S = \frac{1}{4\pi}\int d^2x\sqrt{\hat{g}}\big((\partial\phi)^2 + i\alpha_0\phi\hat{R} + \mu e^{-2ib\phi}(x) + \tilde{\mu}e^{\frac{2i}{b}\phi}(x)\big)$$

here $\hat{R}$ - scalar curvature , Gauss-Bonnet theorem

$$\frac{1}{4\pi}\int dx^2\hat{R}\sqrt{\hat{g}} = 2(1-h) \tag{5.23}$$

for the sphere $h = 0$, i.e., we can identify $\hat{R}\sqrt{\hat{g}} = 8\pi\delta^2(x - R_\infty)$,i.e. this term corresponds to inserting a vertex operator $V_{-\alpha_0}(R_\infty) = e^{-2\pi i\alpha_0\phi(R_\infty)}$; then the correlators remain invariant under the shift $\phi \to \phi + a$ (which preserves the action), if the modified charge conservation condition (5.15) is valid at any given order of perturbation theory.

Literature: [16], [17].

6. Locality and OPE coefficients

On a manifold with no boundaries, like the plane, the only physical operators - the 2d local fields $\phi_{h,\bar{h}}(z,\bar{z})$, now $\bar{z}$ is the compex conjugate. All differential operators apply to the full correlators. We have to combine the chiral holomorphic and anti-holomorphic parts.

• Chiral correlators - multivalued functions. Starting with the 2-point functions - $z = z_{21} = 0$ is a branch cut point and the analytic continuation, monodromy, along a contour around $z = 0$ adds phases $e^{-2\pi i\epsilon 2h}$ with sign depending on the orientation of the contour. Apparently we can compensate these phases multiplying with the anti-holomorphic 2-point function with $\bar{z} =$ complex conjuagate - if the difference of the two dimensions $2h - 2\bar{h}$ is an integer. In fact we want a stronger condition (euclidean locality) that the correlators of **bosonic** fields are invariant under exchange of the fields, hence we need

$$h - \bar{h} = s \in \mathbb{Z}$$

i.e., the spin, to be integer (or half-integer) for fermionic fields. So we should look at possible combinations with this property. This rule is sufficient to ensure locality for the 3-point functions as well. We shall discuss for the time being only scalar fields with $h = \bar{h}$.

• What happens with the 4- point functions? Sketch of the argument:

Recall our fundamental example - we found that the solution of the eqn is an arbitrary linear combination of two independent solutions - with prefactors - powers of $z \sim z_{21}$, times a hypergeometric function - a power series in z.

• Hence to ensure again trivial monodromy of the 2d correlator - sufficient to take diagonal linear combination of the products of holomorphic and anti-holomorphic elementary solutions. This ensures in fact the stronger locality property, the symmetry under the exchange of two operators, $z_1 \leftrightarrow z_2$, or the ratio $z \to \frac{z}{z-1}$, using some relations for the hypergeometric functions.

• Next we have to check the exchange of other pairs of points, say $z_2 \leftrightarrow z_3$, $z \to 1/z$, or any other, however this is not easy: we need another basis of hypergeometric solutions so that the monodromy around the other singular points $z = \infty, z = 1$ is determined just by the overall power. And it is known that the hypergeometric eqn admits different solutions defined in different regions, obtained by analytic continuation - in general a matrix transformation.

• The result of this new representation - effectively different chiral blocks - but - non-diagonal coupling - hence require a constraint on the arbitrary coefficients in the initial linear combination. Constraint of locality - also called "crossing symmetry".

Let analyse in a detail this procedure for a similar example with vertex operators, and derive independently those transformations.

From now on we shall skip the screening charges, assuming that $V_\alpha(z,\bar{z})$ denotes the 2d interaction field. The 2-point function, normalised to $1 = C(\alpha_0-\alpha,\alpha)$ is

$$\langle V_{\alpha_0-\alpha}(x_2)V_\alpha(x_1)\rangle = |z_{21}|^{-2h}$$

Next 3-point or OPE coefficient of 2d fields; consider e.g. the fundamental field

$$V_{b/2}(1)\,V_\beta(0) = C_+V_{\beta+b/2}(0) + C_-V_{\beta-b/2}(0) = \sum_\pm C_{b/2\,\beta}^{\beta\pm b/2}\,V_{\beta\pm b/2}$$

we have

$$C_+ = C(\alpha_0-\beta-b/2, b/2, \beta) = 1\,, \tag{6.1}$$

while to find C_- we need to compute the 2d integral

$$\int d^2x_0 (x_{01}^2)^{-2b\beta}(x_{02}^2)^{-b^2} = \pi (x_{12}^2)^{b(\alpha_0-2\beta)}\,\frac{\gamma(1-2b\beta)\gamma(1-b^2)}{\gamma(2-2\beta-b^2)}\,,\ \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \tag{6.2}$$

hence

$$C_- = C(\alpha_0-\beta+b/2, b/2, \beta) = -\mu\,\pi\frac{\gamma((2\alpha-\alpha_0)b)}{\gamma(b^2)\,\gamma(2\alpha b)} \tag{6.3}$$

NB: More is needed to prove that these are the only possible CVO related to the fundamental field of dimension $h(b/2)$. E.g., if we had Q_- as a screening charge, instead we get computing the analogous to (6.1) integral, a factor $\gamma(2)$ in the OPE constant, which vanishes due to a singularity of the Gamma function in the denominator. This reflects the fact that $Q_-|\frac{b}{2}\rangle = |\frac{b}{2}+\frac{1}{b}\rangle$ is a singular vector (of dimension $h(\frac{b}{2})+2$) which automatically decouples. The same effect has $Q_+^2|\frac{b}{2}\rangle = |\frac{b}{2}-2b\rangle = |\alpha_0-(\frac{b}{2}+\frac{1}{b})\rangle$; both correspond to the level 2 singular vector in the Verma module $V(h(\frac{b}{2}),c)$, and thus are in the kernel of the Felder BRST operator Q_2 in the Fock module and as well in the kernel of its dual in the Fock module

$$Q_2|\tfrac{b}{2}\rangle = |h_{-2,1}=h_{2,-1}\rangle = 0 = \tilde{Q}_1|\tfrac{b}{2}\rangle$$

• Replacing in the 3-point function $b/2$ with arbitrary α s.t. $\beta+\alpha=\alpha_0+b$ leads to the same computation. In particular the first two fusion terms $\alpha+\beta\,,\ \alpha+\beta-b$ are determined again by zero or 1 integral. We shall consider this more general case. The general 3-point OPE coefficients are given by a multiple 2d integrals with the integrand determined by (5.22).

The 4-point function of scalar fields given by one 2d integral reads according to the 2d Coulomb gas prescription

$$\begin{aligned}&\langle e_0 - \alpha_2 - \alpha_3 - \alpha_1 + b | V_{\alpha_3}(1) V_{\alpha_2}(x) | \alpha_1 \rangle \\ &= (x_3^2)^{-h_{123}^4} (x^2)^{2\alpha_2\alpha_1} ((x-1)^2)^{2\alpha_2\alpha_3} \int d^2y ((y-x)^2)^{-2b\alpha_2} (y^2)^{-b^2} (y-1)^2)^{-2b\alpha_3}\end{aligned} \tag{6.4}$$

It can be rewritten as a linear combination of products of chiral integrals.

Instead of performing this computation let us analyse the possible output in terms of contour integrals. We can choose the two contours along pairs of singular points, e.g.

$$\begin{aligned}W_1 &= \lim_{z_4 \to \infty} z_4^{2h_4} \prod_{i>j} z_{ij}^{2\alpha_i\alpha_j} \int_{z_3}^{z_4} dw (z_4 - w)^{-2b\alpha_4} \prod (w - z_i)^{-2\alpha_i b} \\ &= (\frac{z_{21}}{z_{31}})^{2\alpha_1\alpha_2} (\frac{z_{32}}{z_{31}})^{2\alpha_2\alpha_3} z_{31}^{2\alpha_1\alpha_2 + 2\alpha_1\alpha_3) + 2\alpha_2\alpha_3 - 2b(\alpha_1+\alpha_2+\alpha_3)+1} \\ &\int_1^\infty dw (w-1)^{-2b\alpha_3} (w - \frac{z_{21}}{z_{31}})^{-2b\alpha_2} w^{-2b\alpha_1} \\ &\text{with } z = \frac{z_{21} z_{34}}{z_{31} z_{24}} \\ W_1 &= z^{2\alpha_1\alpha_2} (1-z)^{2\alpha_1\alpha_3} z_{31}^{-h_{123}^4} \; I_1(z) =: z_{31}^{-h_{123}^4} G_1 \\ W_2 &= z^{2\alpha_1\alpha_2} (1-z)^{2\alpha_1\alpha_3} z_{31}^{-h_{123}^4} \; I_2(z) = z_{31}^{-h_{123}^4} \, G_2\end{aligned} \tag{6.5}$$

$$\begin{aligned}I_1(\alpha_3, \alpha_2, \alpha_1; z) &= \int_1^\infty dw (w-1)^{-2b\alpha_3} (w-z)^{-2b\alpha_2} w^{-2b\alpha_1} \\ &= \frac{\Gamma(2b(a_2+\alpha_3+\alpha_1)-1)\Gamma(1-2b\alpha_3)}{\Gamma(2b(\alpha_2+\alpha_1)} \, {}_2F_1(2b\alpha_2, 2b(\alpha_1+\alpha_2+\alpha_3)-1; 2b(\alpha_2+\alpha_1); z) \,,\end{aligned}$$

$$\begin{aligned}I_2(\alpha_3, \alpha_2, \alpha_1; z) &= \int_0^z dw (1-w)^{-2b\alpha_3} (z-w)^{-2b\alpha_2} w^{-2b\alpha_1} \\ &= z^{1-2b(\alpha_1+\alpha_2)} \int_0^1 dw (1-zw)^{-2b\alpha_3} (1-w)^{-2b\alpha_2} w^{-2b\alpha_1} \\ &= z^{1-2b(\alpha_1+\alpha_2)} \frac{\Gamma(1-2b\alpha_2)\Gamma(1-2b\alpha_1)}{\Gamma(2-2b(\alpha_1+\alpha_2))} {}_2F_1(2b\alpha_3, 1-2b\alpha_1; 2-2b(\alpha_1+\alpha_2); z)\end{aligned} \tag{6.6}$$

- In the case $\alpha_1 = \frac{b}{2}$ the two chiral correlators reproduce precisely the two solutions (3.15) of the hypergeometric eqn (3.14) - but with parameter further restricted by the charge conservation condition $\alpha_{1234} = \alpha_0 + b = 1/b$; i.e., the Coulomb gas construction provides special solutions of the singular vector decoupling equation.

- Combining (6.4) and (6.6) the overall powers of z are

$$
\begin{aligned}
h(\alpha_2^{(+)}) &= h(\alpha_2 + \alpha_1) - h(\alpha_2) - h(\alpha_1)\,, \\
h(\alpha_2^{(-)}) &= h(\alpha_2 + \alpha_1 - b) - h(\alpha_2) - h(\alpha_1)\,.
\end{aligned}
\tag{6.7}
$$

The integrals in (6.6) are generically multivalued functions of z and the analytic continuation around the point 0 (one of the branch cut points), leads to a multiplication by a phase $e^{2i\pi h(\alpha_2^{(\pm)})}$ - the first comes from the overall power in front of the integral along the contour independent of the coordinates z_1, z_2, and hence is the same as for the free field vertex OPE (5.12). The monodromy of the pair of chiral correlators is diagonal. Similarly, "half" of this transformation, $B_{21}^{(\pm)}$, interpreted as an exchange of the operators $V_{\alpha_1}(z_1)$ and $V_{\alpha_2}(z_2)$, along a positive (anti-clockwise), or negative (clockwise) directions, changes the chiral correlators by a corresponding phase $e^{i\pi h(\alpha_2^{(\pm)})}$, i.e.

$$
B_{21}^{(\epsilon)} = \mathrm{diag}(e^{i\pi\epsilon h(\alpha_2^{(+)})}, e^{i\pi\epsilon h(\alpha_2^{(-)})})
$$

On the other hand the corresponding anti-chiral correlators for $\bar{z}_i$ - the compex conjugates of z_i change by the opposite phases so that the product is invariant, in particular the monodromy is trivial. In a bosonic euclidean correlator locality corresponds to symmetry, i.e., the physical correlator should not change under exchange of any two operators. Since the monodromy of the blocks when z_2 is moving around z_1 is represented by a diagonal matrix we have to take a diagonal combination of products of left and right blocks to construct a 2d correlator of local fields

$$
\begin{aligned}
&G(\alpha_4, \alpha_3, \alpha_2, \alpha_1; z, \bar{z}) = (|z|^2)^{-2\alpha_2 b}(|1-z|^2)^{-2\alpha_3 b} \sum_i X_i I_i(z) \bar{I}_i(\bar{z}) \\
&= \frac{(|1-z|^2)^{-2\alpha_3 b}}{(|z|^2)^{h(\alpha_2)+h(\alpha_1)}} \sum_{\pm} (|z|^2)^{h(\alpha_2^{(\pm)})} C(\alpha_4, \alpha_3, \alpha_\pm) C(\alpha_0 - \alpha_\pm, \alpha_2, \alpha_1) |{}_2F_1(A_\pm, B_\pm; C_\pm; z)|^2
\end{aligned}
\tag{6.8}
$$

In the case $\alpha_1 = b/2$ the parameters in the hypergeometric functions are recovered from (6.6). We have used that $C(0, \alpha, e_0 - \alpha) = 1$ and

$$
V_{\alpha_2}(x_2) V_{\alpha_1}(x_1) \sim \sum_{\alpha} C(\alpha_0 - \alpha, \alpha_2, \alpha_1) V_\alpha(x_1)
$$

Locality (or crossing symmetry) implies that upon exchanging $V_{\alpha_3}(x_3)$ and $V_{\alpha_1}(x_1)$ (or $V_{\alpha_3}(x_3) \leftrightarrow V_{\alpha_2}(x_2)$ we should have

$$G(\alpha_4,\alpha_3,\alpha_2,\alpha_1;z,\bar{z}) = G(\alpha_4,\alpha_1,\alpha_2,\alpha_3;1-z,1-\bar{z}) = |z|^{-4h_2} G(\alpha_4,\alpha_2,\alpha_3,\alpha_1;\frac{1}{z},\frac{1}{\bar{z}}) \tag{6.9}$$

Then the monodromy of the finction of z around the points 1 and infinity resp. is diagonal. To compute the r.h.s. we have to express each of the hypergeometric functions as a linear combination of another pair of solutions of the hypergeometric eqn,

$$_2F_1(A_s,B_s;C_s;z) = F_{s,s'}\ _2F_1(A'_{s'},B'_{s'};C'_{s'};1-z) \tag{6.10}$$

This relation can be taken from the books, or alternatively, derived, using the integral representation and deforming the contour C,

Steps:

Deforming the contour above and below the real axis ($z^a \to (e^{\pi i}z)^a$ anti-clockwise)

$$I_{C_{z_3,\infty}} = -e^{-s\pi i2b\alpha_3} I_{C_{z_2,z_3}} - e^{-s\pi i2b(\alpha_2+\alpha_3)} I_{C_{z_1,z_2}} - e^{-s\pi i2b(\alpha_1+\alpha_2+\alpha_3)} I_{C_{-\infty,z_1}}\,,\ s=\pm$$

$$\Rightarrow\ \sin\pi i2b(\alpha_2+\alpha_3) I_{C_{z_3,\infty}} = -\sin\pi i2b\alpha_2\, I_{C_{z_2,z_3}} + \sin\pi i2b\alpha_1 I_{C_{-\infty,z_1}}\,,$$

$$I_1(\alpha_3,\alpha_2,\alpha_1;z) =$$

$$\frac{\sin\pi i2b\alpha_1}{\sin\pi i2b(\alpha_2+\alpha_3)} I_1(\alpha_1,\alpha_2,\alpha_3;1-z) - \frac{\sin\pi i2b\alpha_2}{\sin\pi i2b(\alpha_2+\alpha_3)} I_2(\alpha_1,\alpha_2,\alpha_3;1-z)\,,$$

$$I_{C_{z_1,z_2}} = -e^{s\pi i2b\alpha_2} I_{C_{z_2,z_3}} - e^{s\pi i2b(\alpha_2+\alpha_3)} I_{C_{z_3,\infty}} - e^{-s\pi i2b(\alpha_1} I_{C_{-\infty,z_1}}\,,$$

$$\Rightarrow\ -\sin\pi i2b(\alpha_2+\alpha_3)\, I_{C_{z_1,z_2}} = \sin\pi i2b\alpha_3\, I_{C_{z_2,z_3}} + \sin\pi i2b(\alpha_1+\alpha_2+\alpha_3) I_{C_{-\infty,z_1}}$$

$$I_2(\alpha_3,\alpha_2,\alpha_1;z) =$$

$$-\frac{\sin\pi i2b(\alpha_1+\alpha_2+\alpha_3)}{\sin\pi i2b(\alpha_2+\alpha_3)} I_1(\alpha_1,\alpha_2,\alpha_3;1-z) - \frac{\sin\pi i2b\alpha_3}{\sin\pi i2b(\alpha_2+\alpha_3)} I_2(\alpha_1,\alpha_2,\alpha_3;1-z)\,,$$

$$I_i = \sum_j \alpha_{ij} I'_j \tag{6.11}$$

This rewrites as a linear transformation of the hypergeometric functions, i.e., determines the coefficients $F_{s,s'}$ in (6.10) comparing with (6.6)

Inserting (6.11) in (6.8) we get three functional equations for the coefficients $X_i(\alpha_3,\alpha_2,\alpha_1)$ and $X_i(\alpha_1,\alpha_2,\alpha_3)$, in particular from the non-diagonal contribution, which has to vanish, as needed to reproduce the diagonal expression in the r.h.s., of (6.9), we determine the ratio of the initial coefficients X_1/X_2. In our case the result is in fact already known, since taking into account the charge conservation condition we determine both OPE constants in the last line in (6.8) from (6.1), or

$$\begin{aligned} X_1 &= \frac{\sin \pi 2b\alpha_3 \sin \pi(2b(\alpha_1+\alpha_2+\alpha_3)-1)}{\sin \pi 2b(\alpha_1+\alpha_2)}\,, \\ X_2 &= \frac{\sin \pi 2b\alpha_1 \sin \pi 2b\alpha_2}{\sin \pi(2-2b(\alpha_1+\alpha_2))}\,. \end{aligned} \tag{6.12}$$

This result is alternatively obtained comparing with the 2d volume integral representation. Note that locality and monodromy invariance are explicit in the volume integral Coulomb gas representation of the (scalar) fields.

- On the other hand we can consider the system of equations in the more general case choosing $\alpha_1 = b/2$, but not imposing the charge conservation condition dictated by the Coulomb gas construction. In that case only the fundamental constants $C(\alpha_0 - (\alpha_2 + s\frac{b}{2}), \alpha_2, \frac{b}{2}) = C_s$ are known, given by (6.1), while the general constant $C(\alpha_4, \alpha_3, \alpha_2 + s\frac{b}{2})$ is unknown and is a subject of the constraints coming from locality. In particular from the vanishing of the non-diagonal term in the transformed function we get an equation for this unknown constant

$$\frac{C(\alpha_4,\alpha_3,\alpha_2+\frac{b}{2})}{C(\alpha_4,\alpha_3,\alpha_2-\frac{b}{2})} = -\frac{C_-}{C_+}\frac{F_{-,+}F_{-,-}}{F_{+,-}F_{+,+}}\,, \tag{6.13}$$

Here $C_\pm$ are the fundamental OPE constants given in (6.1), and the coefficients $F_{s,s'}$ in the transformation (6.10) of the hypergeometric functions, with the parameters of ${}_2F_1(a,b;c;z)$ in (6.6) replaced by ($\alpha_2 = b/2$)

$$a = 2b\alpha_2 + b(\alpha_{1234} - 1/b)\,,\ b = 2b\alpha_{123} - 1 - b(\alpha_{1234} - 1/b)\,,\ c = 2b(\alpha_1+\alpha_2) \tag{6.14}$$

The functional equation (6.13) can be solved imposing as initial condition

$$C(\alpha_1\,,\,\alpha_2\,,\,\alpha_3) = 1 \ \text{ for } \ \sum_i \alpha_i = \alpha_0\,. \tag{6.15}$$

The solution is compactly written as ratio of products of the special function $\Upsilon_b(x)$ expressed in terms of the double Gamma functions $\Gamma_b(x)$.

- An alternative derivation of the Coulomb gas constant is given by considering the multiple integral representation for the 3-point conformal blocks and using the volume integral version of the Selberg formula. Similarly the chirally factorised representation of the 4-point function is generalised by a basic set of multiple contour integrals.

NB: The equation (6.13) (but not the solution) admits an analytic continuation to the region of $c > 25$, i.e., $b \to ib$, and was originally [18] proposed and solved precisely in that region: the Virasoro theory in that region is interpreted as a quantisation of the Liouville theory, to be discussed below

6.1. Summary and comments

1. Different bases for the holomorphic conformal blocks - related by linear invertible transformations implementing analytic continuation.

2. Chiral factorisation of the physical correlation functions; duality (euclidean locality) - the physical correlation functions are independent of the (simultaneous change of) basis of chiral blocks.

3. Primary field OPE coefficients determined from this requirement.

Comments:

1. Coulomb gas does not lead automatically to vanishing of the 2d 3-point correlation functions inconsistent with the fusion rules - this in fact allows to extend the theory and introduce fields covariant under quantum groups $U_q(sl(2))$. The point is that multiple contour integrals of the basic set may become linearly dependent (correspond to the contribution of points related by the reflection maps s_0, s_1 in (2.9)) - reflected in indecomposable quantum group representations. In the local correlators of the minimal theory their contribution is compensated and the fusion rules are restored, taking into account also relations for the braiding matrices.

2. Finding a generalisation of the Selberg integral formula, or its 2d version, is still an unsolved, important and interesting problem (partial generalisations); requires higher rank generalisation of the special functions $\Gamma_b(x)$ and $\Upsilon_b(x)$; see [19] for a recent review of the Selberg formula and its various generalisations and applications.

Without such a generalisation the sl(2) related conformal theories - the minimal Vir and the sl(2) WZW (and some simple, typically level 1 WZW cases) remain the only ones with explicitly known OPE coeffs.

Literature: [16], [20].

7. Braiding and fusing

Recall that the CVO is an intertwining operator

$$V_{\alpha\,\beta}^{\gamma}(z) : F_\beta \to F_\gamma \tag{7.1}$$

and hence is determined by its matrix element between states in the two spaces,

$$z^{h_\alpha+h_\beta-h_\gamma} \langle\gamma|V_{\alpha\,\beta}^{\gamma}(z)|\beta\rangle\rangle\,, \qquad \beta \in F_{\beta,\alpha_0}\, \gamma \in F_{\gamma,\alpha_0} \tag{7.2}$$

Here α, β, γ - in general correspond to states in the representation spaces not necessarily primary, but we can restrict to primary states since the contribution of the descendants in the OPE of primary fields (or of their descendants) is controlled by the symmetry. This matrix element is nonzero only if the three representations satisfy the fusion rules and we can also look at the CVO as a map

$$F_\gamma^* \otimes F_\beta \otimes F_\alpha \to \mathbb{C}$$

i.e., an element of $\mathrm{Hom}(F_\gamma^* \otimes F_\beta \otimes F_\alpha, \mathbb{C})$.

- In general the CVO appear with multiplicities and the matrix element (7.2) will be given not just by a constant, but by some tensor $t_{\alpha\beta}^{\gamma}$ describing the invariants in a tensor product of representations of some finite dimensional algebra, which plays the role of the projective transformations; e.g., it will be the subalgebra of the affine KM algebra for fields described by representations of KM algebra. We shall denote the space of CVO by $\mathcal{V}_{ij}^k \subset \mathcal{V}_k^{*0} \otimes \mathcal{V}_i^0 \otimes \mathcal{V}_j^0 \to \mathcal{C}$ - it has dimension $\mathcal{N}_{ij}^k$.

- Let us now return to the transformation (6.11) which we computed, and which describes the exchange of the two CVO $V_{\alpha_3\ \alpha}^{\beta}(z_3)$ and $V_{\alpha_1 0}^{\alpha_1}(z_1)$ in a 4-point correlation function. The transformation can be rewritten, accounting also for the phase change of the prefactors, as a matrix (from now on α is an index labelling the representations of the chiral algebra, not the Coulomb gas charges)

$$V_{\alpha_3\ \alpha}^{\beta}(z_3)\, V_{\alpha_2\,\alpha_1}^{\alpha}(z_2)\, V_{\alpha_1 0}^{\alpha_1}(z_1)|0\rangle = \sum_{\alpha'} \hat{B}_{\alpha\ \alpha'}^{(\epsilon)} V_{\alpha_1\,\alpha'}^{\beta}(z_1) V_{\alpha_2\,\alpha_3}^{\alpha'}(z_2)\, V_{\alpha_3 0}^{\alpha_3}(z_3)|0\rangle \tag{7.3}$$

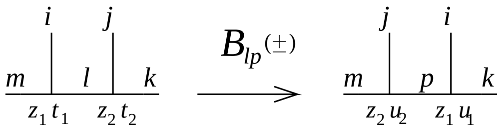
There are two such transformation (labelled by the sign ϵ), depending on the orientation of the contours along which the analytic continuation is done. We can redo the transformation in elementary steps exchanging neighbours only

$$B^{(31)} = \hat{B} = B_{21} B_{32} B_{21}$$

where notation refers to the fixed three ordered points, not to the labels of the concrete interchanged operators. In the example (6.11) α, α' take two values, in general - dictated by the possible fusion channels of the CVO.

The notation here refers to the position of the two adjacent CVO. Another notation for the elementary exchange matrices, i.e., exchanging neighboring vertices, is

$$V_{\alpha_3\,\alpha}^{\alpha_4}(z_3)\, V_{\alpha_2\,\alpha_1}^{\alpha}(z_2) = B_{\alpha\,\alpha'}^{(\epsilon)} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} V_{\alpha_2\,\alpha_1}^{\alpha}(z_2) V_{\alpha_3\,\alpha}^{\alpha_4}(z_3)\,, \quad \epsilon = \text{sign Im } z_{32} \tag{7.4}$$



This is called "non-abelian statistics", anyons, etc. In general each vertex carries an index running from 1 to the multiplicity $\mathcal{N}$, so the exchange matrix depends on these 4 indices (on the picture denoted by t_1, t_2, u_1, u_2) and there is an additional summation. The inverse transformation reads

$$B_{\alpha\,\alpha'}^{(\epsilon)} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} B_{\alpha'\,\beta}^{(-\epsilon)} \begin{bmatrix} \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_1 \end{bmatrix} = \delta_{\alpha,\beta}\,.$$

In particular

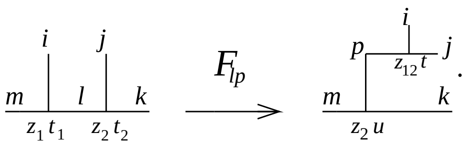
$$B_{\beta\,\beta'}^{(\epsilon)} \begin{bmatrix} \alpha_2 & \alpha_1 \\ \alpha & 0 \end{bmatrix} = \delta_{\beta,\alpha_1} \delta_{\beta',\alpha_2} e^{\pi i \epsilon (h_\alpha - h_1 - h_2)} \tag{7.5}$$

which applies to both B_{21} in (7.3); in the left one (7.5) is applied with $\alpha_2 \to \alpha_3\,, \alpha_1 \to \alpha_2$. From (7.3), (7.5) we compute the middle exchange matrix B_{23}, which unlike the composite one in (7.5), exchanges neighbouring operators

$$B_{\alpha\,\alpha'}^{(\epsilon)} \begin{bmatrix} \alpha_3 & \alpha_1 \\ \beta & \alpha_2 \end{bmatrix} = e^{\pi i \epsilon (-h(\alpha) - h(\alpha') + h_{12} + h_{23})} \hat{B}_{\alpha,\alpha'}^{(31),\epsilon} \tag{7.6}$$

• We can interpret this transformation differently, as an OPE, using the fact that the r.h.s. has well defined behaviour for $z_2 \to z_3$ ($z \to 1$)

$$
\begin{aligned}
&V^{\beta}_{\alpha_3\,\alpha}(z_3)\,V^{\alpha}_{\alpha_2\,\alpha_1}(z_2)\,|\alpha_1\rangle \\
&\qquad\qquad = \sum_{(\alpha';P)\in F_{\alpha'}} F_{\alpha\,\alpha'}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}\langle(\alpha',P)|V^{\alpha'}_{\alpha_3\,\alpha_2}(z_{32})|\alpha_2\rangle\; V^{\alpha_4}_{\alpha'\,\alpha_1}(z_2)|\alpha_1\rangle \\
&\qquad\qquad \sim \sum_{\alpha'} F_{\alpha\,\alpha'}\begin{bmatrix}\alpha_3 & \alpha_2\\ \beta & \alpha_1\end{bmatrix}\frac{c(\alpha',\alpha_3,\alpha_2)}{z_{32}^{h(\alpha_2)+h(\alpha_3)-h(\alpha')}}V^{\beta}_{\alpha'\,\alpha_1}(z_2)\,|\alpha_1\rangle
\end{aligned}
\tag{7.7}
$$



The matrix F - *fusing*, or crossing matrix, graphical notation of Moore -Seiberg (MS) [21].

• The two transformations (7.3) and (7.7) coincide up to a phase -precisely compensates the braiding of the two vertices in the MS block

$$
\begin{aligned}
F_{\alpha\,\alpha'}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} &= e^{-\pi\epsilon(h_4-h_1-h_2-h_3)}\,\hat{B}^{(31,\epsilon)}_{\alpha,\alpha'} \\
&= e^{\pi i\epsilon(h(\alpha)+h(\alpha')-h_2-h_4)}B^{(\epsilon)}_{\alpha\,\alpha'}\begin{bmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{bmatrix}, \\
F^{-1}_{\alpha\,\alpha'}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} &= F_{\alpha\,\alpha'}\begin{bmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{bmatrix} \\
&\text{and} \\
F_{\alpha\,\alpha'}\begin{bmatrix}\alpha_3 & 0\\ \alpha_4 & \alpha_1\end{bmatrix} &= \delta_{\alpha,\alpha_1}\delta_{\alpha',\alpha_3}
\end{aligned}
\tag{7.8}
$$

In the same way as F was computed in the example, it can be computed from the transformation of the general multiple contour basis (originally DF computed one row and one column of this matrix). These matrices have symmetries under certain exchanges of the four indices - they are recovered from the drawings.

• **Completeness** - B and F are invertible linear transformations

$$B : \oplus_p V^s_{ip}\otimes\mathcal{V}^p_{jk}\to\oplus_p\mathcal{V}^s_{jp}\otimes V^p_{ik}$$

$$F : \oplus_p V^s_{ip}\otimes\mathcal{V}^p_{jk}\to\oplus_p\mathcal{V}^s_{pk}\otimes V^p_{ij}$$

Braiding implies commutativity of fusion multiplicities

$$\mathcal{N}_i \mathcal{N}_j = \mathcal{N}_j \mathcal{N}_i$$

i.e. the commutativity of the integer valued matrices N_j is a necessary condition for the existence of B. This can be interpreted as commutativity of an abstract algebra with structure constants N_{ij}^s, while fusing requires

$$\mathcal{N}_i \mathcal{N}_j = \sum_s \mathcal{N}_{ij}^s \mathcal{N}_s \tag{7.9}$$

which can be interpreted as associativity of that algebra - fusion algebra - special case of C (character) -algebra.

7.1. Polynomial equations

Since the exchange of the two CVO can be done in two different ways, i.e., we require associativity, we have the **hexagon** relation (Yang-Baxter type relation)

$$B_{21} B_{32} B_{21} = B_{32} B_{21} B_{32} \tag{7.10}$$

and

$$B_{43} B_{21} = B_{21} B_{43}$$

• Interpretation: n-point chiral correlators - realise representations of the braid group B_n of n elements (strands), with relations satisfied by the generators

$$\{e_i\,, i = 1, \dots n-1\} \qquad e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1}\,, \quad e_i e_j = e_j e_i\,,\ j \neq i \pm 1 \tag{7.11}$$

- exchange matrices also called braiding matrices.

On the plane (Riemann sphere) with n holes - additional constraint on the braid group representation:

$$(e_{n-1} \dots e_2 e_1)^n = \prod_{i=1}^n e^{-2\pi i h_i}\,, \quad e_1 \dots e_{n-1}^2 \dots e_1 = e^{-4\pi i h_1} \tag{7.12}$$

NB: the element in the l.h.s. of the first equality generates the center of the braid group.

The phases in (7.12) are reproduced using the representation in terms of the F matrix, we have

$$\begin{aligned} e_1e_2e_1 &= e^{i\pi\epsilon(h_4-h_1-h_2-h_3)}F \\ e_3e_2e_1 &= e^{-2\pi i\epsilon h_1}F \\ e_1e_2e_3 &= e^{-2\pi i\epsilon h_1}F \end{aligned} \tag{7.13}$$

Also $(e_1e_2e_1)^3 = \prod_{i=1}^n e^{-2\pi i h_i}$.

In a general n-point function with $n > 4$ the basic hexagon eqn (7.10) is nontrivial; for $n = 4$ it takes a simpler form since some of the matrices act diagonally and it can be written equivalently as a relation for the fusing matrices and the phases, or

$$e^{i\pi\epsilon(h_\alpha+h_{\alpha'}-\sum_i h_{\alpha_i})}\, F^{(\epsilon)}_{\alpha\,\alpha'}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \sum_\beta\; F_{\alpha\,\beta}\begin{bmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{bmatrix} e^{i\pi\epsilon h_\beta}\; F_{\beta\,\alpha'}\begin{bmatrix}\alpha_2 & \alpha_3\\ \alpha_4 & \alpha_1\end{bmatrix} \tag{7.14}$$

• Another basic identity satisfied by the matrices F derives from the associativity of the OPE: pentagon eqn - for the derivation see the analogous picture in section 16. Symbolically

$$\sum FFF = FF \tag{7.15}$$

NB: The F transformation plays two roles in the pentagon equation: to relate the two bases of decomposition of a product of three CVO and to define the OPE expansion itself - compare with group theory where the Clebsch-Gordan coefficients (3j - symbols) define the decomposition of a product of two representations, while the 6j symbols relate the two bases in the decomposition of three representations as a consequence of associativity. We will see that in a theory with a boundary these two roles will be played by two different transformations.

The pentagon relation can be looked as a recursive eqn for the general F, once we know the simplest such transformation reflecting the OPE of a fundamental field - the latter fundamental $F's$

$$F_{\alpha_1\pm\frac{b}{2}\;\alpha_3\pm\frac{b}{2}}\begin{bmatrix}\alpha_3 & \frac{b}{2}\\ \alpha_4 & \alpha_1\end{bmatrix}$$

are computed from the transformation of the pair of solutions of the hypergeometric eqn with general parameters (6.14); it is given my a ratio $\frac{\Gamma\Gamma}{\Gamma\Gamma}$ of Γ - functions.

Gauge transformation - normalisation $n(k,i,j)$ of CVO :

$$V^k_{ij} \to n(k,i,j)\; V^k_{ij}\,, \quad F \to \frac{nn}{nn}F$$

preserves the pentagon eqn.

NB: In a proper gauge the F- matrices in the rational Vir models $\alpha = jb - j'/b$ - related to a product of 6j-symbols of quantum groups - a pair of $U_q(sl(2))$ with $q = e^{2\pi i b^2}$ and $q = e^{2\pi i/b^2}$. In the rational case the deformation parameter is a root of $1 = q^N$. - special and we shall restrict to it. Note that the generic $c > 25$ Vir theory also allows to introduce and compute braiding transformations, this time - for a continuous series of Vir representations.

- Besides the the hexagon and the pentagon, there is a further relation involving B and F which comes from duality on the torus, to be discussed later.

- **Locality** condition in general

$$\sum CCFF^{+} = C'C' \tag{7.16}$$

With proper normalisation of the CVO the exchange matrix B can be made unitary; in our previous notation $\alpha'_{ij} = \sqrt{\frac{X_i}{X'_j}}\,\alpha_{ij}$, so that F is unitary too (in this example real matrix elements), to be denoted $F^{(0)}$

- Physical operators (the contribution of the primaries)

$$\Phi_{(a,\bar{a})} = \sum_{f,\bar{f};b\bar{b}} d^{F}_{AB} V^{f}_{a\,b}(z) \otimes \overline{V}^{\bar{f}}_{\bar{a}\,\bar{b}}(\bar{z}) \tag{7.17}$$

d - OPE coeffs, related to former C up to the normalisation of the CVO; if we choose ${d_{IJ}}^{K} = \delta_{i\bar{i}}\delta_{j,\bar{j}}\delta_{k,\bar{k}}{\mathcal{N}_{ij}}^{k}$ in the diagonal case, then

$${C_{AB}}^{F} = {d_{AB}}^{F}\sqrt{C^{f}_{ab}{C_{\bar{a}\bar{b}}}^{\bar{f}}}$$

and (7.16) becomes

$$\begin{aligned} &\sum_{f,\bar{f}} F^{(0)}_{f\,p} F^{(0)*}_{\bar{f}\,p'} = \delta_{p,p'} \\ &\sum_{f,\bar{f}} d_{DCF}\, {d_{BA}}^{F}\; F^{(0)}_{f\,t} F^{(0)*}_{\bar{f}\,\bar{t}} = d_{DAT}\, {d_{BC}}^{T} \end{aligned} \tag{7.18}$$

In general the physical space

$$\mathcal{H} = \oplus Z_{ij}\mathcal{H}_i \otimes \overline{\mathcal{H}}_j\,, \quad Z_{ij} \in \mathbb{Z}_{\geq 0} \tag{7.19}$$

contains non-scalar operators, integer spin $h - \bar{h}$ for bosons, appearing with some multiplicities Z_{ij}. They will be determined from another consistency condition - *modular invariance* of partition functions on the torus, to be discussed below.

- The fact that F is unknown in general - raises the problem of finding alternative more algebraic way of determining the (relative) OPE coefficients in higher rank cases.

Literature: [21], see also [22], [23], [24], [25].

8. Affine Kac-Moody algebras and WZW models

We shall consider more general realisations of the Vir algebra in terms of "elementary" currents, the so called Sugawara construction, with currents taking values in a non abelian Lie algebra. There will be no "deformation" by an additional term, so this is an analog of the $c = 1$ abelian case, however the compexity of the current algebra itself - equivalently the nontrivial lagrangian behind, the WZW model, ensures that we get a non-trivial CFT, in particular with generically non-trivial Vir central charge. This justifies a brief review of the basic notions of the affine Kac - Moody algebras and some info on their (unitary) representations.

8.1. Warming exercise - Weyl group of $sl(n)$ - recovered from the braid group B_n

In the last lecture we mentioned the braid group B_n of n elements (strands) with $n-1$ generators satisfying the Artin relations (7.11). Let us now impose the additional constraint that the squares of all generators are trivial, $e^2 = 1$. Thus the braiding of neighbouring strands is replaced by just a transposition, and the group is reduced to the group of permutations S_n. Let us see how it is realised.

Consider an euclidean n-dim space with scalar product $<\,,\,>$ and an orthogonal basis $< l_i, l_j >= \delta_{ij}$ $\{l_i\,,\ i = 1, 2, \ldots n\}$. Introduce the set of vectors, called simple roots

$$\Pi = \{\alpha_i - l_i - l_{i+1}\,, i = 1, 2, \ldots, n-1\} \tag{8.1}$$

They have norm 2 and their scalar products are $< \alpha_i, \alpha_{i\pm1} >= -1$, or $< \alpha_i, \alpha_j >= 0$ for difference of indices $|i - j| \geq 2$. Consider the invertible maps (reflections) in the euclidean space

$$w_{\alpha_i}(\lambda) = \lambda - < \lambda, \alpha_i > \alpha_i$$

which preserve the scalar product. Applying it to the initial orthogonal basis we see that w_{α_i} exchanges l_i, l_{i+1} and that the compositions of such maps satisfy the Artin relations and thus form a (finite) group, the group S_n of permutations consisting of $n!$ elements. What happens with the set Π? We have that w_{α_i} applied on α_i sends it to $-\alpha_i$ and furthermore creates new combinations of the type $l_i - l_j$, $i \neq j$, exhausting all of them. They are thus all written as linear combinations (with coefficients ±1, of identical sign, or 0) of the simple roots. We call the ones with positive coefficients - positive roots. This enlarged set, to be called root system $\triangle$, respectively ($\triangle_+$ - positive root subsystem) of cardinality $|\triangle| = n^2 - n$, contains with any root α_i also the root $-\alpha_i$ and so splits into two subsets; the positive root system contains the simple roots. By construction the permutation group preserves the root system. Furthermore some of the elements of the group - words made of the simple relections can be equivalently realised as reflections given by the formula (8.1) with an arbitrary root and all such reflections w_α for any root are identified as compositions of simple ones.

Example: $n = 3$ - 6 roots, 3 positive roots $\alpha_1, \alpha_2, \alpha_1+\alpha_2$, and correspondingly 3 reflections, $w_1, w_2, w_1 w_2 w_1 = w_{\alpha_1+\alpha_2}$; there are also 2 compositions of reflections $w_1 w_2\,, w_2 w_1$ (not themselves reflections) and together with the identity element **1** we get the 6 element group S_3.

Now suppose that we add another element α_0 to the set of simple roots and postulate that it has norm 2 and nonzero scalar product (equal to -1) with α_1 and α_{n-1} and zero otherwise: it needs and extension of the initial space and norm, we consider now the S_n as abstract group. We get this way a cyclic set of points representing the set of simple roots, two vertices connected with a link if the square of their scalar product is 1. Already on the examples of $n = 2$ (freely generated group of two generators with no relation) and 3 (the Artin relation extended) we see that this addition generates an infinite group. The enlarged set of simple roots describes the set of simple roots of the affine algebra $\hat{sl}(n)$ and the infinite group is its affine Weyl group.

8.2. Basics on the affine Kac-Moody algebras, integrable representations

Affine KM Lie algebra associated with a finite dimensional simple Lie algebra $\overline{\mathfrak{g}}$ of rank r: these algebras possess a non-degenerate invariant symmetric bilinear form, the Killing-Cartan form.

Consider the ring $\mathbb{C}(t, t^{-1})$ of Laurent polynomials in a variable t. Then $\mathbb{C}(t, t^{-1}) \otimes \overline{\mathfrak{g}}$ is a Lie algebra with commutator $[X^a \otimes t^n, X^b \otimes t^m] = [X^a, X^b] \otimes t^{n+m}$ - the loop algebra, which for $|t| = 1$ is defined by the maps $S^1 \to \overline{\mathfrak{g}}$. This algebra is centrally extended by an element $\hat{k}$

$$\begin{aligned}
&\mathfrak{g} = \mathbb{C}(t, t^{-1}) \otimes \overline{\mathfrak{g}} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}d = \mathfrak{g}' \oplus \mathbb{C}d\,, \\
&[X^a \otimes t^n, X^b \otimes t^m] = [X^a, X^b] \otimes t^{n+m} + n\,\delta_{n+m,0}\ < X^a, X^b >\ \hat{k} \\
&\text{or, with } X^a \otimes t^n = X^a_n\,, \\
&[X^a_n, X^b_m] = f^{ab}{}_c X^c_{n+m} + n\,\delta_{n+m,0}\ < X^a, X^b >\ \hat{k}\,,
\end{aligned} \tag{8.2}$$

and another element d (derivation $t\frac{d}{dt}$) is added

$$[d, X_n^a] = nX_n^a\,;$$

in (8.2) $< X^a, X^b >= q^{ab}$ is the Killing-Cartan form in $\bar{\mathfrak{g}}$.

The zero mode subalgebra of $\mathfrak{g}$ (the derivation d acts trivially and is skipped) with structure constants ${f^{ab}}_c$ reproduces this *horizontal* algebra $\bar{\mathfrak{g}}$. The Cartan (i.e. maximal abelian, diagonalised) subalgebra of $\mathfrak{g}$ is that of $\bar{\mathfrak{g}}$ with the two elements added,

$\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}\bar{k} \oplus \mathbb{C}d$.

The Killing-Cartan form remains non-degenerate when resricted to the Cartan subalgebra $\bar{\mathfrak{h}}$ and hence induces an isomorphism $\bar{\mathfrak{h}} \to \bar{\mathfrak{h}}^*$ and a nondegenerate form on $\bar{\mathfrak{h}}^*$. The normalisation is chosen so that the long roots have length 2. It is extended to $\mathfrak{g}$ by

$$< X_n^a, X_m^b >= \delta_{m+n,0} < X^a, X^b >\,, \; < k, d >= 1$$

all the rest are vanishing; preserves the invariance $< [X,Y], Z > + < Y, [X,Z] >= 0$ of the form (a consequence of the cyclicity of the trace in the initial definition). The element d - *derivation* , added to make the extended K-C form non-degenerate. (More generally, the affine algebra can be extended to a semidirect product with a Witt algebra of derivations" $d_i = -t^{i+1}\frac{d}{dt}$ so that $-d_0$ is identified with L_0.)

- Weights $\lambda \in h^*$

$$\lambda = \bar{\lambda} + k\Lambda_0 + n\delta\,, \; i = 1, 2, \ldots, r \tag{8.3}$$

where weights dual to to the central charge and to d respectively, are introduced, $\Lambda_0(\hat{k}) = 1 = \delta(d)$, and $\bar{\lambda} \in \bar{\mathfrak{h}}^*$ is the horizontal projection of λ, or

$$\lambda(z_i h^i + t\hat{k} + \tau d) = z_i \bar{\lambda}(h^i) + tk + \tau n$$

Furthermore the symmetric bilinear form on $\bar{h}^*$ is extended to h^* by

$$(\bar{h}^*, \mathbb{C}\delta + \mathbb{C}\Lambda_0) = 0; (\delta, \delta) = (\Lambda_0, \Lambda_0) = 0\,; \; (\delta, \Lambda_0) = 1$$

so that

$$\lambda = ((\lambda, \alpha_i^\vee), (\lambda, \delta), (\lambda, \Lambda_0)) = (\lambda(h^i)\,,\, \lambda(\hat{k})\,,\, \lambda(d))$$

and more generally

$$(\lambda, \lambda') = (\bar{\lambda}, \bar{\lambda}') + k_\lambda n_{\lambda'} + k_{\lambda'} n_\lambda$$

(In what follows both notations $< \cdot, \cdot >$ and $(\cdot, \cdot)$ are used.)

Denoting by $\bar{\Lambda}_i$ the fundamental weights $\bar{\Lambda}_i$ of $\bar{\mathfrak{g}}$ we can write

$$\bar{\lambda} = \sum_{i=1}^{r} (\bar{\lambda}, \alpha_i^\vee) \bar{\Lambda}_i \,, \quad \bar{\lambda}(h^i) = (\bar{\lambda}, \alpha_i^\vee) \,.$$

Here $\alpha_i^\vee$ are the dual roots (coroots) $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. The algebras $\bar{\mathfrak{g}}$ with $\alpha_i^\vee = \alpha_i$ and hence $\alpha_i^2 = (\alpha_i, \alpha_i = 2$ for all roots - simply laced - of type (complex algebras) $A_{n-1} = sl(n)\,, D_r = so(2r)\,, E_6, E_7, E_8$. Cartan matrix $C_{ij} = (\alpha_i, \alpha_j^\vee) = 2\frac{(\alpha_i,\alpha_j)}{\alpha_j^2}$, not symmetric in general; visualised by the Dynkin diagram: to every simple root - a node (colored if two different lengths), nodes $i \neq j$ joint by $C_{ij}C_{ji}$ edges, can be $1, 2$, or 3.

- **Roots of $\mathfrak{g}$** :

they also determine root decomposition of the algebra: roots are assign to raising and lowering generators

roots are: real and imaginary,

imaginary - of multiplicity r- correspond to $H_n^i = H^i \otimes t^n$, $n \neq 0$, H^i - Cartan generators of $\bar{\mathfrak{g}}$;

real positive - correspond to weight vectors $E_0^\alpha\,, E_n^{\pm\alpha}\,, n > 0$ in the Cartan-Weyl basis;

simple - correspond to $E_0^{\alpha_i}\,, i = 1, 2, \ldots, r\,,$ $E_1^{-\theta}$, θ - maximal root

$$\begin{aligned} \triangle &= \{n\delta + \alpha \,|\alpha \in \overline{\triangle}\,, n \in \mathbb{Z}\} \cup \{n\delta \,|n \in \mathbb{Z}\,, n \neq 0\} = \triangle^{\text{re}} \cup \triangle^{\text{im}} \\ \triangle_+^{\text{re}} &= \overline{\triangle}_+ \cup \{n\delta + \alpha \,|\alpha \in \overline{\triangle}\,, n \in \mathbb{Z}_{>0}\} \end{aligned} \tag{8.4}$$

Simple roots of $\mathfrak{g}$:

$$\Pi = \{\alpha_i \in \overline{\Pi}, i = 1, 2, \ldots, r\,; \alpha_0 = \delta - \theta\}$$

recall max root θ of $\bar{\mathfrak{g}}$, $\theta = \sum_{i=1}^r a_i \alpha_i = \sum_{i=1}^r a_i^\vee \alpha_i^\vee$, $h^\vee = \sum_{i=1}^r a_i^\vee + 1 = \sum_{i=0}^r a_i^\vee$.

Cartan matrix $((\alpha_i, \alpha_j^\vee))_{i,j=0}^r$ of affine algebra, where coroots $(n\delta + \alpha)^\vee = \frac{2}{\alpha^2}(n\delta + \alpha)$; Dynkin diagram.

- **Triangular decomposition**

$$\mathfrak{g} = n_+ \oplus \mathfrak{h} \oplus n_- \,, \quad \mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}d\,, \quad n_+ = \bar{n}_+ + \sum_{k>0} (t^k \otimes \bar{\mathfrak{g}})$$

Example:

$\hat{sl}(2) = A_1^{(1)} \,: \Pi = \{\alpha\,,\ \alpha_0 = \delta - \alpha\}\,,\ < X^0, X^0 >= 2 = 2 < X^+, X^- >$

$$
\begin{aligned}
&[X_n^+, X_m^-] = X_{n+m}^0 + n\delta_{m+n,0}k\,,\\
&[X_n^0, X_m^\pm] = \pm 2X_{m+n}^\pm\,,\ [X_n^0, X_m^0] = 2k\delta_{m+n,0}\,,\\
&n_+ = l.s.\{X_n^a\,,\ n>0\,, X_0^+\}\,,\text{generated by } \{\mathbf{e} = X_0^+\,, \mathbf{e_0} = X_1^-\}\,,\\
&n_- = l.s.\{X_{-n}^a\,, n>0\,, X_0^-\}\,,\ \text{generated by } \{\mathbf{f} = X_0^-\,, \mathbf{f_0} = X_{-1}^+\}\,,\\
&\mathfrak{h} = \{\mathbf{h} = l.s.(\mathbf{h} = X_0^0\,, \mathbf{h_0} = k - X_0^0\,, d)\}
\end{aligned}
\tag{8.5}
$$

In general (affine Chevalley generators)

$$\mathbf{f_0} := X_{-1}^{\theta}\,,\ \mathbf{e_0} := X_1^{-\theta}\,, \mathbf{h_0} := k - X_0^{\theta}\,,\quad [\mathbf{e_0}, \mathbf{f_0}] = \mathbf{h_0} \tag{8.6}$$

- **Affine Weyl group** W

  generated by simple reflections $w_j = w_{\alpha_j}\,, j = 0, 1, \ldots$; for any root α

$$w_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$$

is a reflection with respect to the hyperplane perpendicular to α

Any element - "word" - a product of simple reflections which satisfy relations

$$w_i^2 = e\,, (w_i w_j)^{m_{ij}} = e$$

where m_{ij} are positive integers or ∞, so that $x^\infty = e$. W preserves the form $(.,.)$, keeps invariant δ and the set of real roots.

- Another representation - by **affine translations** $w = t_\beta w_\alpha\,,\ \beta \in Q^\vee = \sum_{i=1}^r \mathbb{Z}\alpha_i^\vee\,, \alpha \in \overline{\Delta}$, W -semidirect product,

$$W = \overline{W} \ltimes t_{Q^\vee}$$

$$t_{\alpha^\vee} = w_{\delta-\alpha} w_\alpha = w_\alpha w_{\delta+\alpha}\,, \text{for } \alpha \in \overline{\Delta}\,,\ t_\alpha t_\beta = t_{\alpha+\beta}$$

$$t_\beta(\lambda) = \lambda + \langle\lambda\,, \delta\rangle\beta - \Big(\langle\lambda\,,\beta\rangle + \frac{1}{2}\langle\beta\,,\beta\rangle\langle\lambda\,,\delta\rangle\Big)\delta\,,\quad \beta \in Q^\vee \tag{8.7}$$

NB: Note that λ translates by $kQ^\vee$, i.e., the translation depends on the level.

For $\alpha \in \overline{\Delta}\,,\ l\delta - \alpha \in \Delta_+^{\rm re}\,,\ y = t_{\beta'}\bar{y} \in W\,, \beta\,, \beta' \in Q^\vee$ one has the properties

$$t_{l\alpha^\vee} = w_{l\delta-\alpha}\, w_\alpha\,,\quad y\, t_\beta\, y^{-1} = t_{\bar{y}(\beta)}\,,\quad y\, w_\alpha\, y^{-1} = w_{y(\alpha)}\,. \tag{8.8}$$

Horizontal projection

$$\overline{w_0(\lambda)} = \overline{w_\theta t_{-\theta}(\lambda)} = \overline{t_\theta w_\theta(\lambda)} = w_\theta(\bar\lambda) + k\theta$$

- **Fundamental weights,** Weyl vector

$$\begin{aligned}(\Lambda_i, \alpha_j^\vee) &= \delta_{ij}\,, i,j = 0,1,2,\dots,r;\\ \Lambda_i &= \bar\Lambda_i + a_i^\vee \Lambda_0\,, i = 1,2,\dots,r\,,\\ \rho = \sum_{i=0}^{r} \Lambda_i &= \bar\rho + h^\vee \Lambda_0\,,\ (\rho,\delta) = h^\vee\,,\ (\rho,\alpha_j) = 1\,, j = 0,1,\dots,r\,,\end{aligned} \tag{8.9}$$

Weight lattice of $\mathfrak{g}$: $P = \sum_{i=0}^{r} \mathbb{Z}\Lambda_i = \overline{P} + \mathbb{Z}\Lambda_0 + \mathbb{C}\delta$;

$P_+ = \sum_{i=0}^{r} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{C}\delta$ - the set of dominant integral weights (or fundamental Weyl chamber mod $\mathbb{C}\delta$)

NB: The horizontal projections of such weights $\bar\lambda = \sum_{i=1}^{r} \bar\lambda_i \bar\Lambda_i$ with $\bar\lambda_i = (\bar\lambda, \alpha_i^\vee) \in \mathbb{Z}_+$ serve as the h.w. of finite dimensional irreps of $\bar{\mathfrak{g}}$.

- Select a finite, mod $\mathbb{C}\delta$, subset of P_+ at fixed level k :

$P_+^{(k)}$ - the **set of integrable weights**, or, alternatively define for $k \in \mathbb{Z}_{>0}$ in terms of horizontal weights

$$P_+^{(k)} := \{\bar\lambda \in \overline{P} | (\bar\lambda, \alpha_i^\vee) \geq 0\,,\ (\bar\lambda, \theta) \leq k\ \} \tag{8.10}$$

This is a fundamental domain with respect to the action of W - for any $\lambda \in \overline{P} + \mathbb{Z}\Lambda_0$ there exists $w \in W$ s.t. $\overline{w(\lambda)} \in P_+^{(k)}$.

Examples - the alcove of integrable representations of $\hat{sl(2)}_k$ coincides with the root diagram of of $sl(k+1)$. The alcoves for $\hat{sl(2)}_k$ are depicted at the end of section 11 below.

- **Verma module** $V(\lambda) = V(\bar\lambda, k)$ of h.w. λ - freely generated by the subalgebra n_- acting on the h.w. vector $|\lambda\rangle$ - a vector of weight λ

$$n_+|\lambda\rangle = 0\,,\quad X|\lambda\rangle = \lambda(X)|\lambda\rangle\,, X \in \mathfrak{h}\,,$$

and for the h.w. we fix the eigenvalue of $L_0 = -d$ to zero, i.e., $\lambda = \bar\lambda + k\Lambda_0$, $\lambda(L_0) = 0$. The value of the center is the same for all states of the module. For integrable weights the module is graded by the eigenvalues of L_0 and each eigenspace is finite dimensional.

E.g., the zero-grade subspace is isomorphic to the finite dimensional subalgebra $\bar n_-$.

- **Kac-Kazhdan reducibility conditions:**
  Let $k + h^\vee \neq 0$. A Verma module $V(\lambda) = V(\bar\lambda, k)$ is reducible iff

$$(\lambda + \hat\rho, \alpha^\vee) \in \mathbb{Z}_{>0} \text{ for } \alpha \in \triangle_+^{\rm re} \tag{8.11}$$

i.e., $(\lambda+\hat\rho, \alpha) = \frac{\alpha^2}{2} m\,,\ m$ - positive integer, and in this case it contains a Verma submodule generated by a singular vector of weight $w_\alpha \cdot \lambda$ [26]. Here - shifted action of the affine Weyl group W

$$w_\alpha \cdot \lambda = w_\alpha(\lambda+\rho) - \rho = \lambda - m\alpha \tag{8.12}$$

For β coinciding with a simple root the singular vector is given by the power of the corresponding weight vector. The general singular vectors are also known [27].

Example: $\hat{sl}(2)_k\,,\quad \hat\rho = \Lambda_1 + \Lambda_0 = \bar\Lambda_1 + 2\Lambda_0\,, \lambda_J = 2J\bar\Lambda_1 + k\Lambda_0 = J\alpha + k\Lambda_0$
Generically - two series of reducible Verma modules, parametrised by a pair of positive integers (m,n),

$$\begin{aligned}
&\text{for } \beta = (n-1)\delta + \alpha,\\
&(\lambda + \hat\rho, \beta) = (n-1)(k+2) + (2J+1) = m\,; \quad m,n \in \mathbb{Z}_{>0}\\
&\Rightarrow 2J+1 = m - (n-1)(k+2)\,, J = j - j'/b^2\,,\\
&\text{for } \beta = n\delta - \alpha\,,\\
&(\lambda + \hat\rho, n\delta - \alpha) = n(k+2) - (2J'+1) = m\,; \quad m,n \in \mathbb{Z}_{>0}\\
&\Rightarrow 2J'+1 = -m + n(k+2)\,, J' = -1 - J = 1/b^2 - 1 - j + (j'-1)/b^2\,,
\end{aligned} \tag{8.13}$$

here we denoted $1/b^2 = k+2$. **NB**: same parametrisation as in the Vir models, except for $J'_{j,j'=0}$. There are explicit expressions for these singular vectors.

- Now if both $2J = 2j$ and $k - 2J$ are non-negative integers, i.e.

$$0 \le 2j \le k\,,\ 2j, k \in \mathbb{Z}_{\ge 0} \tag{8.14}$$

there are infinitely many singular vectors, e.g., for $\beta = \alpha \to (\lambda+\rho, \alpha) = 2J+1 \ge 1$, and for $\beta = \alpha_0 \to (\lambda + \rho, \delta - \alpha) = k = 2 - 2J - 1 \ge 1$. Then the Verma submodules $V(w_1 \cdot \lambda)$ and $V(w_0 \cdot \lambda)$ are also reducible - etc. The first two of them, corresponding to the two simple roots $\alpha_1\,, \alpha_0$ are given by the monomials

$$(X_0^-)^{2j+1} v_j\,, \quad (X_{-1}^+)^{k+2-(2j+1)} v_j$$

The irreducible representation of such h.w. - integrable representation of the affine algebra, obtained by factoring over the union of the two maximal submodules.

NB: Another two infinite series - admissible representations: rational level $k+2=b^2=p'/p$; parametrisation closely related to that of the minimal Vir modules [28].

Integrable representations in general - irreps with integrable weight $\lambda \in P^+$. Denoting

$$\begin{aligned} X_0^{(-\alpha_i)} &: = \mathbf{f_i}\,, i=1,2,\ldots,r\,, X_{-1}^{\theta} := \mathbf{f_0}\,, \\ \text{singular vectors } &\rightarrow \mathbf{f_i}^{(\lambda+\rho,\alpha_i^\vee)} v_\lambda = 0\,, i=0,1,2,\ldots,r, \end{aligned} \tag{8.15}$$

for the affine reflection the exponent reads $(\lambda+\rho,\alpha_0) = k + h^\vee - (\bar{\lambda}+\overline{\rho},\theta)$.

- Unitarity

  Define a hermitian form with hermitian conjugation of the generators

$$(X_n^\alpha)^+ = X_{-n}^{-\alpha}\,,\ (h_n^i)^+ = h_{-n}^i$$

and dual vacuum state $\langle\lambda|\lambda\rangle = 1$. E.g., norm of the state

$$|X_{-1}^\theta|\lambda\rangle|^2 = \langle\lambda|X_1^{-\theta}\ X_{-1}^\theta|\lambda\rangle = \langle\lambda|(k - X_0^\theta)|\rangle = k - (\lambda,\theta)$$

hence $k-(\lambda,\theta)\geq 0$ is one of the necessary conditions for unitarity.

Literature: [29].

9. Characters of the integrable representations, modular transformations

• We have discussed h.w. irreps of the affine KM algebras characterised by dominant integral weights $\lambda = \bar{\lambda} + k\Lambda_0$ - the integrable, unitary representations of $\mathfrak{g}$. This is a finite set for a fixed value of k, described in terms of the horisontal highest weights by (8.10). The h.w. of the derivation operator d is fixed to zero; the weights of the states then will be $\mu = \bar{\mu} + k\Lambda_0 - n\delta$ with positive n; $[d, X^a_{-n}] = -nX^a_{-n}$.
• The integrable representations arise as factors of maximally reducible Verma modules with integrable highest weights. These Verma modules of $\mathfrak{g}$ are analogs of the reducible Verma modules M_λ of $\bar{\mathfrak{g}}$ which give rise, after factorisation of the maximal submodule $\cup_{\alpha_i \in \bar{\Pi}} M_{w_{\alpha_i} \cdot \bar{\lambda}}$, to the finite dimensional irreps of $\bar{\mathfrak{g}}$. Whence - full analogy with many structures like the characters, where many formulae hold just replacing the finite Weyl group with the infinite affine Weyl group.
• Formal characters - sums of formal exponentials $e^\lambda = e(\lambda)\,,\ \lambda \in P$ - the weight lattice, or elements of the abelian group of translations t_P

$$e^\lambda e^\mu = e^\mu e^\lambda = e^{\lambda+\mu}\,,\ e^0 = 1\,, w(e^\lambda) = e^{w(\lambda)}$$

We can also add them, so they are considered as elements of the group algebra of t_P; algebra multiplication = group multiplication.

The characters of the integrable representations of $\mathfrak{g}$ are generalisation of the finite-dim characters of $\bar{\mathfrak{g}}$, so let us first recall them.

9.1. Characters of representations of finite dimensional algebras

The support of a Verma module $M_{\bar{\lambda}}$ of $\bar{\mathfrak{g}}$ is the set of weights in $\bar{\lambda} - \bar{Q}^+$ with non-zero multiplicity. Its formal character is the generating function of the multiplicities of representation states, described by the Kostant partition function, $\mathrm{mult}_{\bar{\lambda}}(\mu = \bar{\lambda} - \beta) = K_\beta$ - the number of partitions of $\beta \in \bar{Q}_+$ into a sum of positive roots,

$$\mathrm{ch}_{\mathrm{V}(\bar{\lambda})} = e^{\bar{\lambda}} \sum_{\beta \in \bar{Q}_+} K_\beta e^{-\beta} = \frac{e^{\bar{\lambda}}}{\prod_{\alpha \in \bar{\Delta}_+} (1 - e^{-\alpha})} \tag{9.1}$$

Finite dimensional irreps of h.w. $\bar{\lambda} \in \bar{P}_+$ - resolved in terms of reducible Verma modules (Bernstein-Gelfand-Gelfand) resolution) and their characters are given by

$$\bar{\chi}_{\bar{\lambda}} = \sum_{w\in\overline{W}} \mathrm{det(w)\, ch_{V(w\cdot\bar{\lambda})}} = \frac{\sum_{\mathrm{w}\in\overline{\mathrm{W}}} \mathrm{det(w)\, e^{w\cdot\bar{\lambda}}}}{\prod_{\alpha\in\bar{\Delta}_+}(1-\mathrm{e}^{-\alpha})} = \frac{\sum_{\mathrm{w}\in\overline{\mathrm{W}}} \mathrm{det(w)\, e^{w(\bar{\lambda}+\overline{\rho})}}}{\sum_{\mathrm{w}\in\overline{\mathrm{W}}} \mathrm{det(w)\, e^{w(\overline{\rho})}}} = \sum_{\mu\in\bar{\mathrm{P}}} \bar{\mathrm{m}}_{\mu}^{\bar{\lambda}} \mathrm{e}^{\mu}$$

and

$$\bar{m}_{\mu}^{\bar{\lambda}} = \sum_{w\in\overline{W}} \mathrm{det(w) K_{w\cdot\bar{\lambda}-\mu}} \tag{9.2}$$

where $\mathrm{det(w)} = \mathrm{det(w_{i_1} ... w_{i_l})} = (-1)^{\mathrm{l}}$ and in the last expression the sum runs actually in a subset of $\bar{P}$, the weight diagram $\Gamma_{\bar{\lambda}}$. The following symmetries under the actions of the Weyl group - ordinary and shifted, hold

$$\bar{m}_{w(\mu)}^{\bar{\lambda}} = \bar{m}_{\mu}^{\bar{\lambda}}\,, \quad \bar{\chi}_{w\cdot\bar{\lambda}} = \mathrm{det(w)}\bar{\chi}_{\bar{\lambda}}\,, \quad \mathrm{w} \in \overline{\mathrm{W}} \tag{9.3}$$

the last equality is an extension of the def (9.2) beyond the dominant Weyl chamber $\bar{P}_+$ to the whole weight lattice $\bar{\lambda} \in \bar{P}$.

There are recursive formulae for the computation of the multiplicities of the states in a irrep, e.g.,

$$\sum_{w\in W} \mathrm{det(w)} \bar{\mathrm{m}}_{\mu-\mathrm{w}\cdot 0}^{\bar{\lambda}} = \delta_{\mu,\bar{\lambda}} \quad \text{for } \mu \in \Gamma_{\bar{\lambda}} \tag{9.4}$$

Proof: Compare the (last) two formulae for the characters in (9.2), but taken with the shifted action and multiply by the denominator in the Weyl formula

$$\sum_{w\in\overline{W}} \mathrm{det(w)\, e^{w\cdot\bar{\lambda}}} = \sum_{\mathrm{w}\in\overline{\mathrm{W}}} \mathrm{det(w)\, e^{w\cdot 0}} \sum_{\mu\in\Gamma_{\bar{\lambda}}} \bar{\mathrm{m}}_{\mu}^{\bar{\lambda}} \mathrm{e}^{\mu} = \sum_{\mu\in\bar{\mathrm{P}}} \mathrm{e}^{\mu} \sum_{\mathrm{w}\in\overline{\mathrm{W}}} \mathrm{det(w)\, \bar{m}_{\mu-w\cdot 0}^{\lambda}}$$

all points $w \cdot \bar{\lambda}$ in the l.h.s., but one, $w = e$, are beyond the weight diagram, comparing with the term $\mu = \bar{\lambda}$ in the r.h.s. we get for any $\mu \in \Gamma_{\bar{\lambda}}$

$$\delta_{\lambda,\mu} = \sum_{w\in\overline{W}} \mathrm{det(w)\, \bar{m}_{\mu-w\cdot 0}^{\lambda}} = \sum_{\mathrm{w}\in\overline{\mathrm{W}}} \mathrm{det(w)\, \bar{m}_{w\cdot\mu}^{\lambda}}$$

Then the formula allows to compute the multiplicities by induction in the height of $\bar{\lambda} - \mu \in \bar{Q}_+$ (height $\mathrm{ht}(k_i\alpha_i) = \sum_i k_i$). The formula extends to μ beyond the weight diagram as well, so that in the r.h.s appears a sum of Kronecker delta's

$$\sum_{w\in\overline{W}} \mathrm{det(w)\, \bar{m}_{\mu-w\cdot 0}^{\bar{\lambda}}} = \sum_{\mathrm{w}\in\overline{\mathrm{W}}} \mathrm{det(w)} \delta_{\mu,\mathrm{w}\cdot\bar{\lambda}} \tag{9.5}$$

The same recursive formula is valid for the affine case.

Ex 1: apply (9.4) to compute the multiplicities of some sl(3) examples, first computing the set $\{w \cdot 0\,, w \in \overline{W}$, namely $\{-w \cdot 0 = 0, \alpha_1, \alpha_2, 2\overline{\rho}, \alpha_2 + \overline{\rho}, \alpha_1 + \overline{\rho}\}$.

9.2. Affine algebra characters

In the affine case dominant integral weights $\bar{P}_+$ - replaced by set of integrable weights $P_+^{(k)}$, the finite Weyl group $\overline{W}$ - replaced by the affine Weyl group $\to$ Weyl-Kac formula, and now - infinite sums since irreps - infinite dimensional, W-infinite. Formulae look analogously, only in the second equality of (9.2) the non-trivial multiplicity of the (imaginary) roots has to be taken into account

$$\mathrm{ch}_\lambda = \sum_\mu \mathrm{mult}_\lambda(\mu)\; e^\mu = \frac{\sum_{w\in W} \det(\mathrm{w})\, \mathrm{e}^{\mathrm{w}\cdot\lambda}}{\prod_{\alpha\in\triangle_+}(1-\mathrm{e}^{-\alpha})^{\mathrm{mult}\ \alpha}} = \frac{\sum_{w\in W} \det(\mathrm{w})\, \mathrm{e}^{\mathrm{w}(\lambda+\rho)}}{\sum_{\mathrm{w}\in \mathrm{W}} \det(\mathrm{w})\, \mathrm{e}^{\mathrm{w}(\rho)}} \tag{9.6}$$

Accordingly, the generalised Kazhdan partition function K_β for $\beta \in Q_+$ ($Q_+ = \sum_{j=0}^r \mathbb{Z}_{\geq 0}\alpha_j\,, \alpha_j \in \Pi$) is the number of partitions of β into a sum of positive roots, each root counted with its multiplicity.

In particular for $\lambda = 0$ we get Macdonald-Weyl denominator identity

$$\sum_{w\in W} \det(\mathrm{w})\mathrm{e}^{\mathrm{w}(\rho)} = \mathrm{e}^\rho \prod_{\alpha\in\triangle_+} (1-\mathrm{e}^{-\alpha})^{\mathrm{mult}\ \alpha}$$

and for $\hat{sl}(2)_k$ this reads as a relation for two variables $x = e^{-\alpha_0}\,,\ y = e^{-\alpha}$ since the positive roots are: $(n-1)\delta + \alpha = (n-1)\alpha_0 + n\alpha\,,\ n\delta + (n-1)\alpha\,, n\delta = n\alpha_0 + n\alpha$ with $n \geq 1$.

$$\prod_{n=1}^\infty (1-x^n\, y^n)(1-x^n\, y^{(n-1)})(1-x^{n-1} y^n) = \sum_{n\mathbb{Z}} (-1)^n x^{n(n+1)/2} y^{n(n-1)/2}$$

• We can rewrite the characters of the integrable representations using the semidirect product form of the affine Weyl group, i.e.,

$$W = \overline{W} \ltimes t_{Q^\vee}$$

that every element can be represented as a product of affine translation and a finite Weyl group element

$$\sum_{w\in W} \det(\mathrm{w})\mathrm{e}^{\mathrm{w}(\lambda+\rho)} = \sum_{\overline{\mathrm{w}}\in\overline{\mathrm{W}}} \sum_{\alpha^\vee\in \mathrm{Q}^\vee} \mathrm{e}^{\mathrm{t}_{\alpha^\vee}\overline{\mathrm{w}}(\lambda+\rho)} = \mathrm{e}^{\frac{1}{2(\mathrm{k}+\mathrm{h}^\vee)}|\lambda+\rho|^2\delta} \sum_{\overline{\mathrm{w}}\in\overline{\mathrm{W}}} \Theta_{\mathrm{w}(\lambda+\rho)}$$

where, using the definition of the affine translations (8.7) for $\lambda = \bar{\lambda} + k\Lambda_0$,

$$t_\alpha(\lambda) = k\big(\alpha + \frac{\bar{\lambda}}{k} - \frac{\delta}{2}(|\alpha + \frac{\bar{\lambda}}{k}|^2)\big) - \delta\frac{|\bar{\lambda}|^2}{2k}\Big)\,, \qquad \alpha \in Q^\vee \tag{9.7}$$

one gets a generalised theta function

$$\Theta_\lambda = e^{k\Lambda_0} \sum_{\alpha \in Q^\vee} e^{k(\alpha + \frac{\bar\lambda}{k} - \frac{\delta}{2}|\alpha + \frac{\bar\lambda}{k}|^2)} = e^{k\Lambda_0} \sum_{\beta \in Q^\vee + \frac{\bar\lambda}{k}} e^{k(\beta - \frac{\delta}{2}|\beta|^2)}$$

It is convenient to normalise the characters by the prefactor (called modular anomaly) collecting the terms $e^{\cdots\delta}$ so that

$$\chi_\lambda = e^{-m_\lambda \delta} \sum_\mu \mathrm{mult}_\lambda(\mu) e^\mu = \frac{\sum_{w \in \overline{W}} \det(\mathrm{w})\, \Theta_{\mathrm{w}(\lambda+\rho)}}{\sum_{\mathrm{w} \in \overline{\mathrm{W}}} \det(\mathrm{w})\, \Theta_{\mathrm{w}(\rho)}} \tag{9.8}$$

Explicitly the modular anomaly m_λ is given,

$$m_\lambda = \frac{|\lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2h^\vee} \tag{9.9}$$

and using the Freudenthal-de Vries strange formula,

$$|\overline{\rho}|^2 = \frac{h^\vee}{12} \dim g$$

it is rewritten as

$$m_\lambda = \frac{|\lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2h^\vee} = \frac{(\bar\lambda + 2\rho, \bar\lambda)}{2(k+h^\vee)} - \frac{1}{24} \frac{\dim \bar{\mathfrak{g}}\, k}{k+h^\vee} = h^{\mathrm{Su}}_{\bar\lambda} - \frac{c^{\mathrm{Su}}_k}{24}\,, \tag{9.10}$$

The first term in the r.h.s coincides with the eigenvalue of the second Casimir of the horizontal algebra $\bar{\mathfrak{g}}$ on the h.w. state, divided by $k + h^\vee$,

$$\begin{aligned} C^{(2)} = q_{ab} X^a X^b &= \sum_{\alpha \in \bar\Delta_+} (e^\alpha f^\alpha + f^\alpha e^\alpha) + \sum_{i,j=1}^r q_{ij} h^i h^j \\ &= 2 \sum_{\alpha \in \bar\Delta_+} f^\alpha e^\alpha + \sum_{i,j=1}^r q_{ij} h^i h^j + \sum_{\alpha \in \bar\Delta_+} h^\alpha \end{aligned} \tag{9.11}$$

$$C^{(2)} |\bar\lambda\rangle = (\bar\lambda + 2\rho, \bar\lambda) |\bar\lambda\rangle$$

where q_{ab} - inverse of the Killing-Cartan form).

• Up to now we considered the formal character with formal exponentials. Let now evaluate the weights $\mu = \bar\mu + k\Lambda_0 - n\delta$ in (9.8) with non-negative n. Denote by $z = \zeta_i h^i \in \bar{\mathfrak{h}}$ and take an element

$$h = 2\pi i (z + t\hat{k} - \tau d) \in \mathfrak{h}\,, \quad z \in \bar{\mathfrak{h}}\,, t \in \mathbb{C}\,, \tau \in H_+$$

where H_+ is the upper complex plane, Im $\tau > 0$. Evaluating any weight $\mu(h) = 2\pi i(\mu_i\zeta_i + tk + \tau n)$ and denoting $e^{2\pi i\tau} = q$ we get

$$e^{\mu(h)} = q^n e^{2\pi ikt} e^{2\pi i\bar{\mu}_i\zeta_i} \, , \; e^{-m_\lambda\delta(h)} = q^{\text{Su}_{\bar{\lambda}} - \frac{c_k}{24}}$$

Recall that $(-n)$ are eigenvalues of the derivation $d = -L_0$ so that we can write the evaluated character (using the first equality in (9.8)) as

$$\chi_\lambda(\zeta, \tau, t) = e^{2\pi itk} \text{tr}_\lambda q^{L_0 - \frac{c(k)}{24}} e^{2\pi i \sum_{i=1}^r h^i\zeta_i} \tag{9.12}$$

Sometimes also the specialised character $\chi_\lambda(\tau)$ for $\zeta = 0 = t$ is considered - it keeps track only of the eigenspaces of the Vir zero mode L_0.
NB: The specialised character does not distinguish the representation and its conjugated $\chi_\lambda(\tau) = \chi_{\lambda^*}(\tau)$

$$\chi_{\lambda^*}(\zeta, \tau, t) = \chi_\lambda(-\zeta, \tau, t)$$

Def: A h.w. representation of $\bar{\mathfrak{g}}$ has a l.w. state $w^{(0)}(\bar{\lambda})$ where $w^{(0)}$ is the longest element of $\overline{W}$. Define a h.w. representation with a h.w. $\lambda^* := (-w^{(0)})(\bar{\lambda})$ - all weights are with opposite signs compared with the initial rep. of h.w. $\bar{\lambda}$ and thus the sign in front of h^i in (9.12) is inverted. E.g. for $sl(n)$ - reflection of Dynkin labels $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n) \to \bar{\lambda}^* = (\bar{\lambda}_n, \bar{\lambda}_{n-1}, \dots, \bar{\lambda}_1)$ - this is the $\mathbb{Z}_2$ symmetry of the Dynkin diagram.

- The eigenvalues h_λ^{Su} of L_0 and the coefficient c_k^{Su} in the characters (9.12) coincide with the scaling dimensions and the central charge of a Vir algebra realised as an extension of the affine KM algebra - this is a generalisation of the Vir symmetry in the abelian $U(1)$ case considered earlier. This theory, in which primaries are labelled by the finite set of integrable representations of the KM algebra, is known as WZW theory, and will be discussed in more detail later on. It is based on the Sugawara construction

$$T(z) = \frac{:X_a X^a:(z)}{2(k + h^\vee)} \tag{9.13}$$

which leads to the Vir central charge in (9.10) and the eigenvalue of the primary state

$$L_0|\lambda\rangle = \frac{C^{(2)}(X)}{2(k + h^\vee)}|\lambda\rangle$$

in (9.10). In the $sl(2)$ case $\lambda = 2J\bar{\Lambda}_1 + k\Lambda_0$

$$h_\lambda^{\rm Su} = \frac{J(J+1)}{k+2}\,, \quad c_k^{\rm Su} = \frac{3k}{k+2}$$

Example: $\hat{sl}(2)_k$, let $\bar{\lambda} + \bar{\rho} = (2j+1)\bar{\Lambda}_1$, then $\beta \in Q$ is given by $\beta = n\alpha$, $n \in \mathbb{Z}$, and denoting $u = e^{2\pi i \zeta}$

$$\chi_j^{(k)} = \frac{\Theta_{2j+1}^{(k+2)} - \Theta_{-2j-1}^{(k+2)}}{\Theta_1^{(2)} - \Theta_{-1}^{(2)}}$$

$$\Theta_{2j+1}^{(k+2)}(\zeta,\tau,t) = e^{2\pi i t(k+2)} \sum_{n\in\mathbb{Z}+(2j+1)/2(k+2)} q^{(k+2)n^2} e^{-2\pi i \zeta (k+2)n}$$

9.3. Modular transformations of the characters

For τ in the upper half plane H_+, i.e., Im $\tau > 0$, so that $|e^{2\pi i \tau n}| = e^{-\pi n {\rm Im}\ \tau}| < 1$ the evaluated character is given by an uniformly absolutely convergent infinite series.

Consider the linear span of the finite set of evaluated characters χ_λ of integrable representations at a fixed level.

• This space realises a finite dimensional unitary representation of the modular group $SL(2,\mathbb{Z})$, namely there are matrices $M_{\lambda\,\lambda'}$

$$\begin{aligned} \chi_\lambda(\gamma\cdot(\zeta,t,\tau)) &= \sum_{\lambda' {\rm mod}/\mathcal{C}\delta} M_{\lambda\,\lambda'}(\gamma)\ \chi_{\lambda'}(\zeta,t,\tau) \\ \gamma\cdot(\zeta,t,\tau) &= \begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot(\zeta,t,\tau) = (\frac{\zeta}{c\tau+d}, t - \frac{c|\zeta|^2}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) \end{aligned} \tag{9.14}$$

The two generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\,, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$$

which act on τ as

$$\tau \to t+1 \text{ and } \quad \tau \to -1/\tau$$

resp., are represented by the matrices (to be denoted by the same letter)

$$\begin{aligned} T_{\lambda,\lambda'} &= \delta_{\lambda,\lambda'} e^{2\pi i\ m_\lambda} \\ S_{\lambda,\lambda'} \equiv S^{(k)}_{\bar{\lambda},\bar{\lambda}'} &= i^{|\overline{\Delta}_+|}|P/Q^\vee|^{-1/2}\ (k+h^\vee)^{-r/2} \sum_{w\in\overline{W}} {\rm det(w)} {\rm e}^{-\frac{2\pi {\rm i}}{{\rm k+h^\vee}}({\rm w}(\bar{\lambda}+\bar{\rho}),\bar{\lambda}'+\bar{\rho}))} \end{aligned} \tag{9.15}$$

Here m_λ is the modular anomaly (9.10), and $|P/Q^\vee| = \det((\alpha_{\rm i}^\vee, \alpha_{\rm j}^\vee))$; counts number of points of P one finds in a elementary cell of $Q^\vee$.

The symmetric unitary matrix S is called **modular matrix**, (9.15) - Kac-Peterson formula. In particular

$$S^{(k)}_{\bar\lambda,0} = |P/Q^\vee|^{-1/2}(k+h^\vee)^{-r/2} \prod_{\alpha\in\bar\Delta_+} 2\sin\pi\frac{(\bar\lambda+\bar\rho,\alpha)}{k+h^\vee} \geq S^{(k)}_{0,0} > 0 \tag{9.16}$$

In agreement with $S^2(\zeta,t,\tau) = (-\zeta,t,\tau)$ we have $S^2 = C$ where $C_{\lambda,\mu} = \delta_{\lambda,\mu^*}$ is the charge conjugation matrix, and $(ST)^3 = C$. The S matrix satisfies (the upper * means complex conjugation)

$$S^{(k)}_{\bar\lambda,\bar\lambda'} = S^{(k)}_{\bar\lambda',\bar\lambda} = S^{(k)\,*}_{\bar\lambda^*,\bar\lambda'} = S^{(k)\,*}_{\bar\lambda,\bar\lambda'^*} \tag{9.17}$$

NB: If we consider the specialised characters with both $\zeta = 0 = t$ - the modular group is the projective group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ since $\tau \to -1/\tau \to \tau$ and hence the group element $S^2 = -e$ should be identified with the identity $\sim e$. However the S matrix is still the general one, i.e., the representation of $PSL(2,\mathbb{Z})$ is projective.

Note that if we consider the partial specialisation $t=0$, the S modular transformation matrix changes by the factor $e^{-2\pi i(\zeta,\zeta)/\tau}$.

- The derivation of the S modular transformation (9.15) is based on generalisations of the basic Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\frac{a}{2}n^2+2\pi i bn} = \frac{2\pi}{\sqrt{a}} \sum_{m\in\mathbb{Z}} e^{-\frac{2\pi}{a}(m+b)^2} \tag{9.18}$$

by itself obtained from the identity

$$\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x}$$

integrated with $e^{-\frac{a}{2}x^2+2\pi i bx}$.

- The complex parameter $\tau \in H_+$ in the upper half - plane is interpreted as the modular parameter of a torus T^2. The latter is equivalently defined by the plane factored by a 2-dim lattice L

$$T = \mathbb{C}/L\,, \quad L = \{m\omega_1 + n\omega_2 | m,n \in \mathbb{Z}\}$$

the parameter τ is the ratio $\tau = \omega_1/\omega_2$ of the two periods; the periods - linearly independent over the real numbers so have nonzero imaginary part, which can be chosen positive. The basis of the two periods is determined up to a linear transformation preserving the lattice, i.e., in the new basis, the points should have again integer coeffs. Such transformations are described by the group $SL(2,\mathbb{Z})$ - modular group of the torus; it preserves the sign of the imaginary part of the ratio τ.

Summary

• The remarkable modular property of the finite set of characters of integrable representations is analogous to the duality transformations of the holomorphic blocks on the plane (braid group); so interpreted as chiral blocks of the theory on the torus - this transformation is generalised, S matrix - $S(j)$, etc. to 1- or 2-point functions on the torus.

In one of the next lectures we shall consider physical quantities defined on the torus - partition functions, and the independence with respect of the modular group - will lead to constraints (analogous to the duality related to locality) and hence to important info on the physical fields in the theory.

Literature: [30], [29].

10. Modular S matrix, fusion algebra, formulae for the fusion multiplicities

Here are some examples of the S matrix

Examples:

The S matrix of the integrable representations of the affine $sl(2)$ algebra

$$S_{(r-1)\bar\Lambda_1+k\Lambda_0,(s-1)\bar\Lambda_1+k\Lambda_0} = S^{(k+2)}_{r,s} = \sqrt{\frac{2}{k+2}}\,\sin(\pi\frac{rs}{k+2})\,,\ 1\le r,s,\le k+1 \qquad (10.1)$$

Similarly one computes the modular transformation matrix of the Vir characters of the minimal models

$$\begin{aligned} S^{(p,p')}_{(r,r'),(s,s')} &= 2\frac{(-1)^{(r+r')(s'+s)}}{\sqrt{2}}\sqrt{\frac{4}{pp'}}\,\sin(\pi\frac{rs\,(p-p')}{p})\sin(\pi\frac{r's'(p-p')}{p'}) \\ &= \sqrt{2}\sqrt{\frac{4}{pp'}}\,\sin\pi br(2\alpha-\alpha_0)\sin\pi\tfrac{r'}{b}(2\alpha-\alpha_0)\,,\ 2\alpha-\alpha_0 = sb-\tfrac{s'}{b} \\ & 1\le r,s\le p-1\,,\quad 1\le r',s'\le p'-1 \end{aligned} \qquad (10.2)$$

In particular the S matrix (3.8) of the Ising model $p'/p=3/4$ is recovered by

$$S^{(4,3)}_{(r,1),(s,1)} = (-1)^{(r+1)(s+1)}\frac{1}{\sqrt{2}}\sin\frac{3\pi rs}{4}$$

The S matrix for $\hat{sl}(N)_1$:
integrable representations - the identity and the fundamental $\bar\Lambda_J+k\Lambda_0$, $(\bar\lambda,\theta)\le 1$

$$S_{\bar\Lambda_t,\bar\Lambda_j} = \frac{1}{\sqrt{N}}e^{\frac{2\pi itj}{N}}\,,t,j=1,2,\dots,N-1\,,\quad S_{0,\bar\Lambda_j}=\frac{1}{\sqrt{N}}$$

The ratios

$$\chi_0(p)=\chi_N(p)\,,\chi_t(p)=\frac{S_{\bar\Lambda_t,\bar\Lambda_p}}{S_{0,\bar\Lambda_p}} = e^{\frac{2\pi\,itp}{N}}\,,\ t=1,2,\dots N-1$$

coincide with the set of characters of the cyclic group $\mathbb{Z}_N$ which satisfy

$$\chi_t(p)\chi_j(p)=\chi_{t+j}(p)$$

and this is the fusion rule of the integrable representations at level $k=1$.

- In general the ratio of modular matrix elements coincides with the characters of finite dimensional irreps of the horizontal algebra, evaluated at particular rational points in $\bar{\mathfrak{g}}$.

$$\overline{\chi}_{\bar\lambda}(e^{-\frac{2\pi i}{k+h^\vee}h}) = \frac{S^{(k)}_{\bar\lambda,\bar\lambda'}}{S^{(k)}_{0,\bar\lambda'}}\,,\qquad \text{where}\ \ h\in\bar{\mathfrak{g}}\ \text{is s.t.}\ (\bar\lambda+\bar\rho)(h)=(\bar\lambda+\bar\rho,\bar\lambda'+\bar\rho)\,,$$

and we can expect that this fact explains the truncation in the standard decomposition rule.

Let us consider again the case $\widehat{sl}(2)$ - the corresponding ratios $S_{r,s}/S_{0,s}$ of the S matrix of the $\hat{sl}(2)_k$ in (10.1) (shifting the labels so that $(\bar{\lambda},\alpha) = r = 2j$). Since these ratios coincide with the $SU(2)$ characters with $U(1)$ angle $\phi = \frac{\pi s}{k+2}$ they should satisfy the standard $SU(2)$ rule of decomposition of a product of two $SU(2)$ characters

$$\overline{\chi}_{j_1}(\phi)\overline{\chi}_{j_2}(\phi) = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} \overline{\chi}_{j_3}(\phi)$$

However, due to the rationality of the angle ϕ these characters are periodic $\overline{\chi}_j = \overline{\chi}_{j+(k+2)}$ and whenever j_3 reaches the value $2j_3 + 1 = k + 2$ - which is outside of the integrable alcove, the rational character vanishes. Furthermore the pair of characters $(\overline{\chi}_j, \overline{\chi}_{k+2-j-1})$ of h.weights related by a shifted action of w_0 have opposite sign, so whenever both appear in the r.h.s. they cancel each other; note that only one of them is then integrable. Recall that the classical bottom bound is also a result of cancellation - of pairs related by the ordinary shifted Weyl reflection $w \cdot \bar{\lambda}_j = \bar{\lambda}_{-j-1}$ using the symmetry property in (9.3) - for the rational characters it extends to the affine reflection w_0, or equivalently these characters are periodic

$$\overline{\chi}_{\bar{\lambda}+(k+h^\vee)\beta} = \overline{\chi}_{\bar{\lambda}}\,, \quad \beta \in Q^\vee$$

Thus we obtain the fusion rule with a truncated upper classical bound,

$$\frac{S_{j_1,j}}{S_{0,j}}\,\frac{S_{j_2,j}}{S_{0,j}} = \sum_{j_3=|j_1-j_2|}^{\min(j_1+j_2,k-j_1-j_2)} \frac{S_{j_3,j}}{S_{0,j}} = \sum_{j_1,j_2}^{j} N_{j_1 j_2}^{j}\,\frac{S_{j_3,j}}{S_{0,j}} \tag{10.3}$$

which can be also derived using the factorisation of the two singular vectors. Alternatively we compute the fusion coefficients in the r.h.s. (using the unitarity of the S matrix) by the formula

$$\begin{aligned} \mathcal{N}_{r_1 r_2}^{r_3} &= \frac{2}{p}\sum(\sin \pi r t)^2\ \chi_{r_1}(r)\chi_{r_2}(r)\chi_{r_3}(r)\,, \\ \chi_p(r) &= \frac{\sin p\,\phi}{\sin\phi}\,, \phi = \pi r\, t\,, \quad t = 1/p \end{aligned} \tag{10.4}$$

This formula is the discrete analog of the integral formula for the $SU(2)$ characters, with Haar measure $\frac{1}{\pi} d\phi\,(\sin\phi)^2$

$$N_{r_1,r_2,r_3} = \tfrac{1}{\pi}\int_0^{2\pi} d\phi \sin^2\phi\, \prod_{i=1}^{3} \tfrac{\sin(r_i\phi)}{\sin\phi}\,, \tag{10.5}$$

Similarly the corresponding formula for the minimal models reproduces the BPZ fusion rules which we have partially derived from the OPE.

These examples demonstrate a general formula for the fusion rule.

• The famous Verlinde formula for the general fusion multiplicity is a generalisation of (10.4) not only for the higher rank WZW model but for any theory for which one has an S matrix with all the properties listed here. The formula states that the fusion multiplicity matrices $(\mathcal{N}_i)_j{}^k$ are simultaneously diagonalised by the modular matrix S,

$$\mathcal{N}_{ij}{}^k = \sum_{l\in\mathcal{I}} \frac{S_{il}}{S_{1l}} S_{jl} S^*_{kl} \tag{10.6}$$

and the S matrix ratios are their eigenvalues.

These integers give the structure constants of the *fusion algebra* : associative, commutative algebra with identity and involution $i \to i^*$ in the set $\mathcal{I}$ of labels lifted to algebra endomorphism (which implies invariance of the structure constants) the algebra can be realised by the matrices $\mathcal{N}_i = (\mathcal{N}_{ij}{}^k)$

$$\mathcal{N}_i\,\mathcal{N}_j = \sum_k \mathcal{N}_{ij}{}^k\,\mathcal{N}_k\,, \tag{10.7}$$

satisfying $\mathcal{N}_{i^*j^*}{}^{k^*} = \mathcal{N}_{ij}{}^k = \mathcal{N}_{i^*k}{}^j$; in general it is sufficient to postulate $\mathcal{N}_{jt}{}^1 = \delta_{jt^*}$, then from associativity follows that $\mathcal{N}_{it^*}{}^{p^*} = \mathcal{N}_{ip}{}^j$. The ratio of S matrix elements realises the 1-dim representations of the fusion algebra. We will refer to this algebra as Verlinde algebra and will find representations different from the Verlinde one in (10.6).

• On the other hand for the higher rank WZW models one can follow the classical character analogy as in the $sl(2)$ case to arrive at another formula [29], [31] (and show that it reproduces Verlinde formula); the virtue of this formula is that it ensures that $\mathcal{N}_{ij}{}^k$ are integers, though does not prove that they are non-negative integers.

• To see what happens in general recall the multiplication of classical characters.

Generically the irreps in the product of the two irreps λ and β are described by the weights of the shifted by β weight diagram Γ_λ of λ and if some of these weights in $\Gamma_\lambda + \beta \subset \bar{P}$ are beyond the Weyl chamber $\bar{P}_+$ of integral dominant weights, they are brought back into $\bar{P}_+$ by a shifted Weyl action. More precisely, exploiting the two equivalent representations of the formal characters (9.2) - as a sum of formal exponentials t_μ taken with multiplicity $\overline{m}_\mu^\lambda$, or the Weyl formula, one obtains

$$
\begin{aligned}
\overline{\chi}_\lambda\, \overline{\chi}_\beta &= \sum_{\mu' \in \Gamma_\lambda + \beta} \overline{m}^\lambda_{\mu'-\beta} \frac{\sum_{w\in\overline{W}} \det(w) t_{w(\mu'+\rho)}}{\sum_{w\in\overline{W}} \det(w) t_{w(\rho)})} \\
&= \sum_{\gamma\in\bar{P}_+} \Big(\sum_{w'\in\overline{W}} \det(w')\overline{m}^\lambda_{w'\cdot\gamma-\beta}\Big)\overline{\chi}_\gamma = \sum_{\gamma\in\bar{P}_+} \overline{N}_{\lambda\beta}{}^\gamma\, \overline{\chi}_\gamma\,, \\
\overline{N}_{\lambda\beta}{}^\gamma &= \sum_{w\in\overline{W}} \det(w)\overline{m}^\lambda_{w\cdot\gamma-\beta}
\end{aligned}
\tag{10.8}
$$

representing at the second step the weights $\mu' + \rho = w'(\gamma+\rho)$ with $\gamma \in \bar{P}_+$, or $\mu' = w' \cdot \gamma$. We thus recover the tensor product decomposition multiplicities $\overline{N}_{\lambda\beta}{}^\gamma$ - the classical analog $\overline{N}_{\lambda\beta}{}^\gamma$ of the fusion multiplicity $\mathcal{N}_{\lambda\beta}{}^\gamma$. Apparently this does not generalise to the product of the true affine characters, since then level will double.

But as in the sl(2) case we can use the fact that the ratios of S do satisfy the classical formula possibly with cancellations of some terms in the r.h.s.

$$
\begin{aligned}
\overline{\chi}_\lambda\, \overline{\chi}_\beta &= \sum_{\gamma\in\bar{P}_+} \overline{N}_{\lambda\beta}{}^\gamma\, \overline{\chi}_\gamma\,, \\
&= \sum_{\gamma\in P_+^{(k)}} N_{\lambda\beta}{}^\gamma\, \overline{\chi}_\gamma
\end{aligned}
\tag{10.9}
$$

The rational characters are periodic, i.e., invariant under affine translations

$$
\overline{\chi}_{\bar{\lambda}+(k+h^\vee)\beta} = \overline{\chi}_{\bar{\lambda}}\,, \quad \beta \in Q^\vee
$$

or equivalently, the symmetry property in (9.4) extends to the full affine group W. Hence the dominant Weyl chamber $\bar{P}_+$ in (10.8) - is replaced by the alcove of integrable weights P_+^k, which is a fundamental domain with respect to the shifted action of the affine Weyl group in the weight lattice $\bar{P}$ and the sum runs over the affine Weyl group. For the fixed points of this action - the boundaries of the alcove, the periodic character vanishes. For a triple of weights belonging to the alcove $\bar{P}_+^k$ of integrable highest weights (8.10) we get the formula, generalising the classical formula above

$$
\mathcal{N}_{\lambda\beta}{}^\gamma = \sum_{w\in W} \det(w)\, \bar{N}_{\lambda\beta}^{w\cdot\gamma} = \sum_{w\in W} \det(w)\, \bar{m}^\lambda_{w\cdot\gamma-\beta}\,,
\tag{10.10}
$$

and these multiplicities have to be identified with the fusion multiplicities. Here the finite weights are used and the action of W is understood as the projection $\overline{w \cdot (\bar{\lambda} + k\Lambda_0)}$.

- From the relation $(ST)^3 = C$

$$STS = T^{-1}ST^{-1}\,, \text{ or} \qquad S^*T^{-1}S^* = ST^{-1}S = TS^*T$$

combined with Verlinde formula one obtains a relation expressing the S matrix in terms of the Verlinde matrices

$$S_{ij} = S_{11} \sum_m \; e^{2\pi i(\triangle_i + \triangle_j - \triangle_m)} \; d_m \; \mathcal{N}_{ij}{}^m\,. \tag{10.11}$$

and vice versa - Verlinde formula is reproduced from (10.11), which itself is a special case of a more general identity [21] involving the modular matrix $S(j)$ of 1-point correlators on the torus. If we invert the sign in the phases in (10.11) - the result will be S^*. The r.h.s. can be identified, in a certain gauge, with the square e_i^2 of a braiding matrix.

$$\frac{S_{ij}}{S_{11}} = d_i d_j B_{11}^2(+) = d_i d_j \sum_m \; B_{1m}(ji^*)(+)B_{m1}(i^*j)(+)$$

- **Asymptotics** of the (restricted) characters: from

$$\chi_\lambda(\tau) = \sum S^*_{\lambda\mu}\chi_\mu(-1/\tau)$$

the limit $\tau \to i\epsilon$ the dominant term in the r.h.s. comes from the identity rep

$$\lim_{\tau \to i\epsilon} \chi_\lambda(\tau) = \lim_{\tau \to i\epsilon} S_{\lambda 0}\chi_0(-1/\tau) = S_{\lambda 0} q^{\frac{c}{24}}\,, \quad c = \frac{k \dim \bar{\mathfrak{g}}}{k + h^\vee} \tag{10.12}$$

or

$$\lim_{q \to 1} \frac{\mathrm{tr}_{\mathcal{V}_\lambda} q^{L_0 - c/24}}{\mathrm{tr}_{\mathcal{V}_0} q^{L_0 - c/24}} = \frac{S_{\lambda 0}}{S_{00}} = \text{"dimension" of } \mathcal{V}_\lambda$$

- There are other symmetries of the S matrix due to automorphisms of the affine Dynkin diagram.

Literature: [32], [33], [29], [31].

11. Modular invariants, ADE classification, fusion graphs, conformal embeddings

We have seen that on a manifold without boundaries, like a plane or an infinite cylinder, or a torus, there are *two copies* of the chiral algebra, one relative to the holomorphic ("left") coordinates, the other to the antiholomorphic "right" ones. The full Hilbert space of states of the RCFT is assumed to decompose into a finite sum of a product of representations of the chiral algebra

$$\mathcal{H}_P = \oplus Z_{j\bar{j}} \mathcal{V}_j \otimes \overline{\mathcal{V}_{\bar{j}}} \,, \tag{11.1}$$

with some non-negative integer multiplicity $Z_{j\bar{j}}$ for each pair $(j, \bar{j})$, and s.t. $Z_{11} = 1$, reflecting the uniqueness of the physical vacuum. This sum also describes, by the state - field correspondence, the spectrum of the primary fields of the theory.

These nonnegative integers $Z_{j\bar{j}}$ are determined by a consistency condition (Cardy, 1986) [34]. One looks at the partition function of the theory on a torus $\mathcal{T}$ - it is given as a sesquilinear combination of the characters

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}} q^{L_0 - c/24} q^{*\bar{L}_0 - c/24} = \sum_{j,\bar{j} \in \mathcal{I}} \chi_j(\tau)\, Z_{j\,\bar{j}}\, \chi_{\bar{j}}(\tau)^* \tag{11.2}$$

and it is required to be invariant under modular transformations M, $MZM^* = Z$. In particular, the T-invariance fixes the possible spins $s_I = h_i - \bar{h}_{\bar{i}}$ to integers.

[Interpretation in statistical mechanics : We start from a system on a finite cylinder mapping $z = e^{\frac{2\pi w}{l}}$ with further periodic condition $w = w + l' + il$ making the cylinder a finite strip. The partition function is then given by the trace of an evolution operator $e^{-l'H - \bar{l}'\bar{H}}$; for real l', $H + \bar{H}$ is the Hamiltonian, generator of "time" translations and $e^{-t(H+\bar{H})}$ is the transfer matrix on an infinite cylinder. The eigenvalues of H for each conformal block are those of the shifted L_0 according to (4.5) so that we get

$$tr_{\mathcal{H}} e^{-2\pi \frac{l'}{l}(L_0 - \frac{c}{24}) + \frac{\bar{l}'}{l}(\bar{L_0} - \frac{c}{24}))} = tr_{\mathcal{H}} e^{-2\pi(\frac{l'+\bar{l}'}{2l}(L_0 + \bar{L_0} - \frac{c}{12}) + \frac{l'-\bar{l}'}{2l}(L_0 - \bar{L_0}))}$$

the two linear combinations in the r.h.s. give the Hamiltonian and the momentum operator. The continuum limit is taken for infinite l, l' so that $l'/l = -i\tau$, with Im $\tau > 0$, is fixed. The theory could be instead defined starting from a cylinder with period l' and computing the trace of the evolution along l. This is equivalent to replacing τ with $-1/\tau$.

$$tr_{\mathcal{H}} e^{-Im\tau \hat{H} + iRe\tau \hat{P}}\,, \qquad \hat{H} = (L_0 + \bar{L_0} - \frac{c}{12})\,, \hat{P} = (L_0 - \bar{L_0})]$$

• One is thus led to the problem of finding all possible sesquilinear forms (11.2) with non negative integer coefficients that are modular invariant, and such that $\mathcal{N}_{11} = 1$. Since the finite set of characters of any RCFT, labelled by $\mathcal{I}$, supports a unitary representation of the modular group, this implies that any diagonal combination of characters $Z = \sum_{i\in\mathcal{I}} \chi_i(q)\chi_i(\bar{q})$ is modular invariant. In that case, all representations appearing in (11.1) are left-right symmetric, and thus all primary fields are spinless: $h_j - h_{\bar{j}} = 0$. This situation is referred to as the "diagonal case" or "diagonal theory". Other solutions are, however, known to exist. They contain pairs with non-zero integer spin, along with a set of scalars, labelled by a subset $\{j = \bar{j}\}$ of the set $\mathcal{I}$, some appearing with nontrivial multiplicity Z_{jj}; this set will be denoted $\mathcal{E} = \{(j, \alpha | j \in \mathcal{I}, \alpha = 1, 2 \ldots, Z_{jj}\}$, assuming that an index is added to account for these multiplicities.

11.1. The ADE classification

The famous ADE classification [35] of the $\widehat{sl}(2)_k$ related modular invariants - the minimal models and the integrable $\widehat{sl}(2)_k$, is based on the observation that the set $\mathcal{E}$ parametrising the scalar fields in each modular invariant coincides with the set of Coxeter exponents of the corresponding simply laced Lie algebra of Coxeter number $h = k + 2$. Thus one can associate with any modular invariant a Dynkin diagram G of type A, D, E.

Notation $1 \le r = 2j+1 \le k+1 = h-1$; in the minimal models the role of the Coxeter number h is taken over by p or p'.

Exponents:

The integers m_i give the degrees minus 1 of a set of algebraically independent generators of the ring of invariant polynomials; $\max\{m_i\}) + 1 = h$, the Coxeter number. They also parametrise the eigenvalues of the Coxeter element of the Weyl group $\overline{W}$ (the product of all simple reflections), which are given by $\{e^{\frac{2\pi m_k}{h}}\}$ with $m_i \in \mathcal{E}$; and $\max\{m_i\}) + 1 = h$.

• The diagonal case corresponds to the algebra $\bar{\mathfrak{g}} = A_{N-1} = sl(N)$ for which $h = N - 1$ and the exponents take the values of the shifted integrable weights for the fixed level k $1 \le m_i = (2j_i\bar{\Lambda} + \bar{\rho}, \alpha) \le h - 1$, i.e., all integers between 1 and $k + 1 = h - 1$. There is a modular invariant for all k an d for odd k these are the only invariants.

• For $h = 4l$ or $h = 2 + 4l$, $l = 1, 2, ...$ - two infinite $D_{h/2+1}$ series, $D_{\text{odd}} = D_{2l+1}$ and $D_{\text{even}} = D_{2l+2}$; the exponent $m = 2l + 1$ in the D_{even} case appears with multiplicity 2.

• For $h = 12, 18, 30$ there are besides the infinite A and D three exceptional modular invariants

$$
\begin{aligned}
A_{h-1}\,, \quad & \mathcal{E} = \{1,2,\ldots,N-1\} = \mathcal{I}\,, \ h = N \\
D_n\,, \quad & \mathcal{E} = \{1,3,5,\ldots,2n-3,n-1\}\,, h = 2n-2 \\
E_6\,, \quad & \mathcal{E} = \{1,4,5,7,8,11\}\,, \ h = 12 \\
E_7\,, \quad & \mathcal{E} = \{1,5,7,9,11,13,17\}\,, \ h = 18 \\
E_8\,, \quad & \mathcal{E} = \{1,7,11,13,17,19,23,29\}\,, \ h = 30
\end{aligned} \tag{11.3}
$$

• Besides the diagonal (scalar) terms each of the non-diagonal modular invariants contains off-diagonal terms describing fields of integer spin: generically they are among the set $\{s_r = h_r - h_{h-r}\, r \in \mathcal{I}\,, r \neq h-r\}$ in all but the three exceptional invariants where also other combinations $s_{r,\bar{r}}$ appear. In the E_6 and E_8 cases all labels belong to the exponent set $\mathcal{E}$,

$$
\begin{aligned}
E_6: \quad & Z = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 \\
E_8: \quad & Z = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2
\end{aligned} \tag{11.4}
$$

The $D_{\rm even}$ invariant has also a block-diagonal form

$$
D_{2l+2}: \quad Z = \frac{1}{2} \sum_{r\in\mathcal{E}\,, r\neq 2l+1} |\chi_r + \chi_{h-r}|^2 + 2|\chi_{2l+1}|^2\,, \qquad 4l+2
$$

• On the other hand the $D_{\rm odd}$ invariant can be described using an automorphism ρ of the set of representations and acting with it on the labels of the left (or right) chiral fields of the diagonal invariant A_{4l-1} of the same level $k+2 = 4l$. Namely, $\rho(r) = h - r$ for even r (half integer isospins $(r-1)/2$), while the odd ones are kept fixed, i.e., $\rho = \sigma$ on the even and $\rho = \mathrm{Id}$ - for the odd, in particular preserves the identity. Since the even labels appear in $\mathcal{N}$ in pairs (conservation of "2-ality") this automorphism preserves the fusion rules.

$$
D_{2l+1}: \quad Z = \sum_{r\in\mathcal{I}} \chi_r \chi^*_{\rho(r)} = \sum_{r\in\mathcal{E}} |\chi_r|^2 + \sum_{r-\mathrm{even}, r\neq 2l} \chi_r \chi^*_{h-r}\,, \qquad h = 4l
$$

Finally the E_7 invariant is closely related to D_{10} - first interpreted as a diagonal modular invariant of blocks labelled by $1,3,5,7,9,9'$, fusion - mod σ, then exchanging the blocks $9' \leftrightarrow [3 \oplus 15]$.....

$$
E_7: \quad Z = |\chi_1 + \chi_{17}|^2 + (\chi_3 + \chi_{15})\chi_9^* + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{13}|^2 + \chi_9(\chi_3^* + \chi_{15}^*) + |\chi_9|^2
$$

The block-diagonal modular invariants can be interpreted as diagonal invs of an extended chiral algebra. Its representations decompose into representations of the initial chiral algebra.

• The classification for the Vir minimal models is closely related and corresponds to pair of graphs (A, G) or (G, A), where G is one of the A, D, E graphs, modulo the $\mathbb{Z}_2$ symmetry, Namely a sum over the second index in $\chi_{r,s}$ is added and a overall 1/2 factor is added to account for the $\mathbb{Z}_2$ symmetry $\chi_{r,s} = \chi_{p-r,p'-s}$.

11.2. The NIMreps

The set $\mathcal{E}$ parametrises the eigenvalues of the Cartan matrix C, or equivalently the eigenvalues $\{\gamma_l = 2\cos\frac{\pi l}{h}\,, l \in \mathcal{E}\}$ of the adjacency matrix $2\delta_{ab} - C_{ab}$ of the graph G. The set of these eigenvalues is a subset (with multiplicity) of the set of the eigenvalues of the fundamental Verlinde fusion matrix $\mathcal{N}_2$ given by

$$2\cos\frac{\pi l}{k+2} = \frac{S^{(k)}_{2\,l}}{S^{(k)}_{1\,l}}$$

• We recall that $\mathcal{N}_2$ is the adjacency matrix of the A_{k+1} graph associated with the diagonal modular invariant; we shall denote the general adjacency matrix $n_2 = 2\delta_{ab} - C_{ab}$. This matrix is diagonalised in a orthonormal basis, i.e., by an unitary matrix ψ^l_a (S in the diagonal case)

$$(n_2)_a^b = \sum_{l\in\mathcal{E}} \frac{S_{2l}}{S_{1l}} \psi^l_a \psi^{l\,*}_b \tag{11.5}$$

Note that the cardinality of the set of vertices $\mathcal{V}$ is the same as that of the exponents, $|\mathcal{V}| = |\mathcal{E}|$. (By an abuse of notation here it is assumed that for coinciding exponents, the projection appears in the S matrix elements, while the eigenvectors are distinguished, i.e., $l = (\omega_l, s) \in \mathcal{E}\,, \omega_l \in \mathcal{I}\,, s = 1, 2\ldots, Z_{\omega_l\omega_l}$ and $S_{ml} = S_{m\omega_l}$.)

• The decompositions of the powers of $\mathcal{N}_2$ generate all the fusion matrices $\mathcal{N}_r\,, r \in \mathcal{I}$. What is essentially used in these products is the product of eigenvalues since they realise 1-dim representations of the Verlinde fusion algebra. But we can do the same starting from the eigenvalues of (11.5). We obtain a set of matrices - as many as $|\mathcal{I}|$, i.e., the number of representations for the given k [36]. These matrices are diagonalised as in (11.5),

$$n_{ra}{}^b = (n_r)_a^b = \sum_{l\in\mathcal{E}} \frac{S_{rl}}{S_{1l}} \psi^l_a \psi^{l,*}_b\,,\ r \in \mathcal{I} \tag{11.6}$$

Apparently they satisfy the relation

$$n_i n_j = \sum_s \mathcal{N}_{ij}{}^s n_s \tag{11.7}$$

By construction these matrices are integer valued. One can prove, using properties of the Coxeter element, that they are actually non-negative integer valued matrices [37]. The relation (11.7) can be interpreted as a non-negative integer valued matrix representation (NIM-rep) of the Verlinde fusion algebra associated with each graph, i.e., with each $\widehat{sl}(2)$ modular invariant.

• The solutions of (11.7) can be interpreted as intertwiners when looked as rectangular matrices, $n_a G = A n_a$, where G and A are the adjacency matrices of the two graphs at the same h.

• Graphs generalizing the ADE Dynkin diagrams, determined by the spectrum of higher rank modular invariants, i.e. satisfying the generalisation of NIM-rep, were also constructed [38] ($sl(N)$ analogs of the D series, see title page of [2]) and [36] (graphs corresponding to the $\widehat{sl}(3)_k$ modular invariants); in particular as solutions of the generalisation of NIM-rep. **NB:** The spectrum itself is not sufficient for higher rank - several isospectral graphs associated with the spectrum of a given modular invariant.

11.3. Relation to the relative scalar OPE coeffs

In general the OPE constants of the non-diagonal (A_{g-1}, D_l), (A_{g-1}, E_l) theories are proportional to the (square roots of the) constants $C_{\alpha\,\beta}^{\gamma}$ in the diagonal theory (A_{g-1}, A_{h-1}) with the same Coxeter number h,

$$D_{AB}^{C} = d_{AB}^{C} \sqrt{C_{\alpha\,\beta}^{\gamma}\, C_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}}} \tag{11.8}$$

where $A = (\alpha, \bar{\alpha})$, etc., and the relative OPE constants $D_{1,2}AB^C$ are set to zero if the fusion multiplicities $\mathcal{N}_{\alpha\,\beta}^{\gamma}$ or $\mathcal{N}_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}}$ vanish.

The relative scalar OPE coefficients $d_{AB}^{C} = d_{(\alpha\alpha)\,(\beta\beta)}^{\quad\;\;(\gamma\gamma)}$ of the non-diagonal theories are expressed in terms of the orthonormalised eigenvectors of the A-D-E Cartan matrix (or of the adjacency matrix $G_{ab} = 2\delta_{ab} - C_{ab}$), $\{\psi^{\alpha} = (\psi_a^{\alpha}), \alpha \in \mathcal{E}\}$. Namely, these constants coincide [39] with the structure constants of the Pasquier algebra [40]

$$d_{(\alpha\alpha)\,(\beta\beta)}^{\quad\;\;(\gamma\gamma)} = M_{\alpha\beta}^{\;\;\gamma}\,, \tag{11.9}$$

where

$$M_{\alpha\beta}^{\;\;\gamma} = \sum_{a\in\mathcal{V}} \frac{\psi_a^{\alpha}\psi_a^{\beta}\psi_a^{\gamma\,*}}{\psi_a^{1}}\,. \tag{11.10}$$

The precise statement is that one can fix the freedom in the signs of both the d's and the M's so that this equality holds. This result was originally almost an empiric observation (based on the lattice ADE models of Pasquier [41]); we shall see that it is derived in the framework of boundary CFT.

11.4. Graph algebra

We can also interpret (as we did with the A) any of the Dynkin diagrams - root graphs, also as fusion graphs.

Namely to any vertex a one associates a matrix $\hat{N}_a = (\hat{N}_a)_b^c$ the identity matrix is associated to a distinguished vertex on the graph denoted by 1, while $\hat{N}_2 = n_2$. these matrices are determined from the action of the adjacency matrix and more generally of the given solution of NIM-rep

$$n_\lambda \, \hat{N}_a = \sum_\gamma n_{\lambda a}{}^b \, \hat{N}_b \,. \tag{11.11}$$

Since the numbers $n_{2a}{}^b$ describe the graph - they are non-zero if the vertices are related, given the graph we can easily compute the action of n_2, etc The new matrices $\hat{N}_a$ are represented by a formula analogous to Verlinde one, but with S replaced by ϕ with the summation running over the exponents

$$\hat{N}_{ab}{}^{\,c} = \sum_{\lambda \in \mathcal{E}} \frac{\psi_a^\lambda \psi_b^\lambda \psi_c^{\lambda\,*}}{\psi_1^\lambda} \tag{11.12}$$

they close another algebra - graph algebra (dual to the Pasquier algebra) and (11.11) asserts that the graph algebra matrices span a module of the fusion algebra. These matrices are nonnegative integer valued in the ADE cases with the exception of E_7 and D_{odd}.

11.5. How to get modular invariants from known ones? (Some examples)

The block-diagonal modular invariants can be interpreted as diagonal invariants of an *extended chiral algebra*. Its representations decompose into representations of the initial chiral algebra with some branching coefficients b^i_λ =mult$_{B_i}(\lambda)$

$$\begin{aligned} \chi_{\mathcal{B}_i} &= \sum_\lambda b^i_\lambda \, \chi_\lambda \,, \\ Z = \sum_{\mathcal{B}_i} |\chi_{\mathcal{B}_i}|^2 &= \sum_i |\sum_\lambda b^i_\lambda \, \chi_\lambda|^2 = \sum_{\lambda,\bar{\lambda}} \chi_\lambda \, Z_{\lambda\bar{\lambda}} \, \chi^*_{\bar{\lambda}} \\ Z_{\lambda\bar{\lambda}} &= \sum_i b^i_\lambda b^i_{\bar{\lambda}} \end{aligned} \tag{11.13}$$

and satisfying the consistency conditions

$$\sum_c S^{\rm ext}_{ac} b^c_\mu = \sum_\lambda b^a_\lambda S_{\lambda\mu} \,, \quad b^a_1 = \delta_{a1}$$

Recall that the scaling dimensions of the fields in a given block differ by integers since the mixed terms describe local non-zero spin fields. In particular the block $B_{\mathbf{1}}$ of the identity representation contains besides the identity itself also representations describing fields with integer dimensions - these fields extend the initial algebra to a bigger chiral algebra.

• So the first step is to describe all possible extensions of the given chiral algebra.

A particular class of such extensions comes from the so called **conformal embeddings** of affine algebras

$$\mathfrak{p}_k \subset \mathfrak{g}_1$$

(each could be simple or semisimple - a direct sum), characterised by identical central charges - which implies that the bigger algebra has level $k = 1$

$$\frac{k \dim \overline{\mathfrak{p}}}{k + h^\vee_{\overline{\mathfrak{p}}}} = \frac{\dim \overline{\mathfrak{g}}}{1 + h^\vee_{\overline{\mathfrak{g}}}}$$

NB: In general the branching functions of the affine chracters depend on τ (there are some infinite series of q), only in the case of conformal embeddings they reduce to numerical coefficients, accounting also for the normalisation with the modular anomaly factors.

Example: $\widehat{sl}(2)_{k=4} \subset \widehat{sl}(3)_1$, 3 integrable representations $\{0, \bar{\Lambda}_1, \bar{\Lambda}_2\}$ with Sugawara dimensions $0, 1/3, 1/3$. Gives rise to the D_4 modular invariant

$$Z = |\chi_1 + \chi_5|^2 + 2|\chi_3|^2$$

The affine currents $J^a(z) = \sum X^a_{-n} z^{n-1}$ correspond to descendants at grade 1 in the identity module of $\mathfrak{g}_1$, $J((0)|0\rangle = X^a_{-1}|0\rangle = t^{-1} \otimes X^a|0\rangle$; these states span a representation isomorphic to the adjoint representation of the horisontal algebra $\overline{\mathfrak{g}}$. So the decomposition of the adjoint representation of $\overline{\mathfrak{g}}$ with respect to the subalgebra $\overline{\mathfrak{p}}$ determines the branching coefficients b^1_λ for $\lambda \neq 1$.

In the example: besides the vacuum h.w. state, $|0\rangle$, there are 8 descendant states corresponding to the $sl(3)$ currents J^a , 3 of them can be identified with the currents of $sl(2)$, and the remaining 5 - with the integrable representation $2j + 1 = 5$ of Sugawara dimension 1, which belongs to the identity block. The primary states of the other two integrable $\widehat{sl}(3)_1$ representations, $\bar{\Lambda}_1, \bar{\Lambda}_2$ correspond to the integrable $\widehat{sl}(2)_4$ representation $2j + 1 = 3$ with the same same Sugawara dimension 1/3.

In general the nonvanishing of b^i_λ implies that the representation $\bar\lambda$ of $\bar{\mathfrak{p}}$ appears at some grade of the $\mathfrak{g}$ module B_i and this implies that the eigenvalues of the Sugawara generators L_0 in the modules of the two algebras differ by an integer i.e, there exists some nonnegative integer n

$$\triangle_i + n = \triangle_\lambda$$

• The conformal embeddings are completely classified [42] and in each concrete case one can work out and find all branching coefficients, and thus recover the modular invariant. However there is no general analytic formula for the branching rules.

• An empiric observation of Di Francesco and Zuber [43]: the branching coefficients are related to the matrix representations of the Verlinde algebra, namely they can be identified with the matrix element of ${n_{\lambda 1}}^{a_i}$ for some vertex $a = a_j$. The set T of all such vertices is related to a subalgebra of the graph algebra $\hat{N}_a$, which is isomorphic to the extended Verlinde algebra. This empiric identification can be adopted as a definition (as is effectively done in the subfactor approach), i.e., ${n_{\lambda 1}}^{a_i}$:=$\mathrm{mult}_{B_i}(\lambda)$ and then it can be used as the initial condition for the equations (11.7), which can be solved recursively.

• Modular invariants from **automorphisms**

If ρ is an automorphism of the extended Dynkin diagram which preserves the modular matrices

$$M_{\rho(i)\,\rho(j)} = M_{ij}$$

then $Z_{ij} = \delta_{i\rho(j)}$ is a modular invariant.

Example: the conjugated "diagonal" modular invariant for $sl(n)$

$$Z(\tau) = \sum_{\lambda\in\mathcal{I}} \chi_\lambda(\tau)\, \chi_{\lambda^*}(\tau)^* \tag{11.14}$$

Here the diagonal terms (corresponding to the scalar fields), i.e., the "exponents" , are described by the real representations $C(\lambda) = \lambda$, i.e., $\mathcal{E} = \{\lambda \in P^k_+ | \bar\lambda = \lambda^* = (\lambda_1, \lambda_2, ..., \lambda_2, \lambda_1)\}$.

E.g., in the case $\widehat{sl}(3)_k$ and even k (odd $k+3$)

$$\mathcal{E} = \{(0,0), (1,1), \ldots, (\frac{k}{2}, \frac{k}{2})\}$$

and one obtains with this spectrum a solution of the NIM-reps equation (11.7), described by a 1d graph [36]: namely the graph of $n^b_{\Lambda_1 a}$ coincides with the fusion graph of the isospin

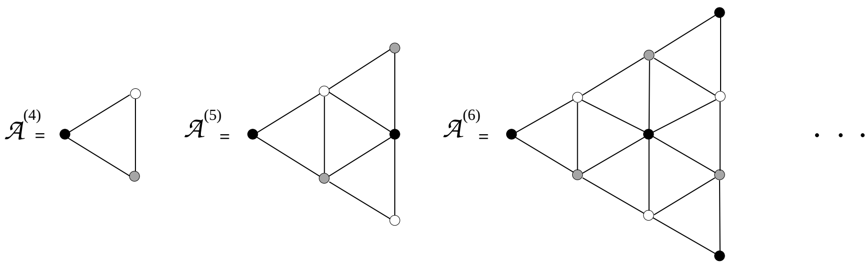
$j = 1$ representation, $\mathcal{N}_{j=1} = \mathcal{N}^2_{j=1/2} - I$, of $\widehat{sl}(2)_{k+1}$, restricted to integer spins, which set describes the vertices of the graph $a = 0, 1, \ldots, \frac{g-3}{2}$

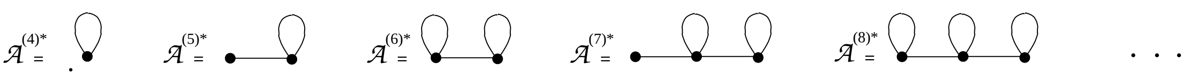
$$n_{\Lambda_1\, a}{}^{b} = \mathcal{N}_{j=1\, a}{}^{b} \tag{11.15}$$

This corresponds to the embedding of $sl(2)$ into $sl(3)$, s.t. the fundamental representation $\bar{\Lambda}_1$ is identified with the 3-dimensional representation of $sl(2)$. The graphs with such adjacency matrix are depicted on the second figure below, namely these are the graphs denoted $\mathcal{A}^{(k+3)^*}$ with odd $k{+}3$. The cases with even $k{+}3$ correspond to another embedding of $sl(2)$ into $sl(3)$, $3 = 2 + 1$, so that

$$n_{\Lambda_1} = 1 + \mathcal{N}_{j=1/2}. \tag{11.16}$$

The first figure represents the diagonal cases, i.e., the graphs of $\mathcal{N}_{\bar{\Lambda}_1}$, which coincide with the integrable alcoves at different k.





Literature: [34], [44], [35], [41], [40], [36], [43], [39], [33], [45], [42].

12. WZW models, Sugawara construction, Knizhnik - Zamolodchikov eqs

We start with the currents $J^a(z)$ with Fourier modes - the generators of the affine KM algebra $\mathfrak{g}$

$$J^a(z) = \sum_n X_n^a z^{-n-1}\,,\ a = 1, 2, ...\dim \bar{\mathfrak{g}}\,, \tag{12.1}$$

with OPE - equivalent to the commutation relations (q^{ab} - Killing -Cartan form)

$$J^a(z_2)J^b(z_1) = \frac{k\, q^{ab}}{z_{21}^2} + \frac{f_c^{ab}\, J^c(z_1)}{z_{21}} + \text{regular}$$

From the OPE, or directly from the algebra, we get

$$[X_n^a, J^b(z)] = f^{ab}{}_c\; z^n\, J^c(z) + k\partial_z z^n q^{ab} \tag{12.2}$$

This can be interpreted as infinitesimal variation of the currents with parameters $\omega^a(z) = \omega_n^a z^n$; similarly for the analogous antiholomorphic commutator

$$\begin{aligned} \delta_\omega J^a(z) &= f^{ab}{}_c\; \omega_b(z)\, J^c(z) + k\partial_z \omega^a(z)\,, \\ \delta_{\bar{\omega}} \bar{J}^a(z) &= f^{ab}{}_c\; \bar{\omega}_b(z)\, \bar{J}^c(z) + k\partial_{\bar{z}} \bar{\omega}^a(z)\,, \end{aligned} \tag{12.3}$$

Classically we can think of the currents $J(z) = t_a J^a(z)$ and $\bar{J}(\bar{z}) = t_a \bar{J}^a(\bar{z})$, where t^a are generators of the (horisontal) finite dimensional Lie subalgebra $\bar{\mathfrak{g}}$, as represented by

$$\begin{aligned} J(z) &= k\, \partial_z g(z,\bar{z})\, g^{-1}(z,\bar{z})\,, \quad \partial_{\bar{z}} J(z) = 0 \\ \bar{J}(\bar{z}) &= -k\, g^{-1}(z,\bar{z}) \partial_{\bar{z}} g(z,\bar{z})\,, \quad \partial_z \bar{J}(\bar{z}) = 0 \end{aligned} \tag{12.4}$$

where $g(z,\bar{z})$ takes values in the group G with algebra $\bar{\mathfrak{g}}$. The transformation

$$g(z,\bar{z}) \to \Omega(z)\, g(z,\bar{z})\, \bar{\Omega}^{-1}(\bar{z}) \tag{12.5}$$

with arbitrary G-valued matrices $\Omega(z) \sim I + \omega(z) = I + t^a \omega_a(z)$ implies the infinitesimal variation of the currents in (12.3) ; the order of g and g^{-1} in the two definitions in (12.4) is essential to reproduce the KM commutators. Note that the conservation of one of the currents implies the conservation of the other.

The question is what is the classical action which is invariant under the factorised gauge transformation (12.5) and thus leads to the conserved currents (12.4)?

12.1. WZNW action

Consider the action [46], [47], with a 2d analog of the 4d Wess-Zumino term

$$\begin{aligned} S_{\lambda,k} &= \frac{1}{4\lambda^2}\int d^2\xi \mathrm{tr}(\partial_\mu g^{-1}\partial_\mu g) + k\Gamma(g) \\ \Gamma(g) &= \frac{1}{24\pi i}\int_B d^3X\epsilon^{\alpha\beta\gamma}\mathrm{tr}(\tilde{g}^{-1}\partial_\alpha\tilde{g}\tilde{g}^{-1}\partial_\beta\tilde{g}\tilde{g}^{-1}\partial_\gamma\tilde{g}) \end{aligned} \tag{12.6}$$

where in the first term the field $g(\xi)$ takes values in some simple compact Lie group G, ξ - coordinates of the 2 dimensional space, e.g. the Riemann sphere. In the second term the integration is over a 3d ball s.t. its boundary ∂B is the 2d space and $\tilde{g} : X \to G$ is an extension of $g(\xi)$,

The boundary value determines the functional $\Gamma(g)$ not uniquely but up to a term of the form $2\pi i\mathbb{Z}$, i.e., $\Gamma(g)$ is multivalued. The difference between two $\Gamma(g)$'s is expressed as an integral of the same type but on a 3d manifold without a boundary, like the sphere S^3 and it is an integer times 2π. E.g., for $G = SU(2)$ the maps $S^3 \to S^3$ are characterised by their winding number, described by the third homotopy class $H_3(G) = \mathbb{Z}$. Hence (12.6) and its exponential in the path integral are unambiguously defined for integer k only. (Locally the integrand is a three form $d\beta$ where β is a 2-form , determined up to a closed one form, so the 3-form is not globally exact.)

- In general this model is massive, solved by the Bethe ansatz, [48], but for $\lambda^2 = 4\pi/k$ the beta function vanishes (infrared stable fixed point of the renorm-group) and the theory becomes massless and conformally invariant. In addition it has the gauge symmetry (12.5) with arbitrary functions of the chiral coordinates: this action $W = kS_{4\pi/k,k}$ is the action usually called WZW (or WZNW) action.

To derive the conservation of the currents one uses the fact that the variation of the density under $g + \delta g$ can be written as a total derivative and so reduces to a 2d functional. The equation of motion then reads as the conservation of anti-holomorphic current $\bar{J}$, which implies the conservation of J.

- KM currents $J(z), \bar{J}(\bar{z})$ - generators of the chiral gauge symmetry, the KM generators - infinitely many conserved currents.

12.2. Quantisation of the WZW model

The fields - composite operators of the group elements g - difficult to construct apart from the particular case when correlators of g itself are considered - this is the case analysed in the original paper of Knizhnik and Zamolodchikov [49]. The WZW action (the action at the fixed renorm-group point) is classically invariant with respect to the infinite dimensional group of conformal coordinate transformations. But in the quantum theory the fields may get anomalous dimensions. Thus the original fields $g(z,\bar{z})$ turn out to acquire anomalous dimension - it corresponds to the fundamental fields (in the sense of KM integrable representations). Similarly the lagrangean density of the first, sigma model, term is interpreted also as having anomalous dimension - a descendant in the adjoint integrable representation.

• Instead of trying to use directly the action and build composite fields - one can follow the general bootstrap scheme, i.e., describe a field realisation of the integrable representations of the KM algebra and exploit all the restrictions coming from the KM and Vir symmetries in order to construct the correlators.

• In particular in this way one can consider even models described by non-integer values of the level k and the weights, i.e., non-integrable models, for which formally the WZW action has no sense.

The vacuum state $|\lambda = 0\rangle$ of the identity irreducible representation is annihilated by all the zero modes $X_0^a|0\rangle = 0$ since the raising generators $X_0^{-\alpha_i}|0\rangle$ are singular vectors in the corresponding reducible Verma module, i.e., the vacuum and its dual are invariant under the finite dimensional subalgebra $\bar{\mathfrak{g}}$.

Primary field transformation - OPE or commutator:

$$[X_n^a, \phi_\lambda(z)] = z^n t^a \phi_\lambda(z) \tag{12.7}$$

Fields transform according to matrix representations of the generators of the finite dimensional Lie algebra $\bar{\mathfrak{g}}$, i.e., they are some $\bar{\mathfrak{g}}$ tensors, analogous to the main term in (12.2) which is the transformation of the adjoint representation. We shall consider in the example of $sl(2)$ another functional realisation without indices, fields $\phi_\lambda(x,z)$ will be polynomials of an auxiliary variable x and then the generators in the r.h.s. of (12.7) will be realised as differential operators similar to the realisation of the $sl(2)$ subalgebra of Vir.

The highest state λ is identified with $\phi_\lambda^{(u)}(0)|0\rangle$ where u is the state of heighest weight in the finite dimensional irrep of $\bar{\mathfrak{g}}$.

From (12.7) one computes the correlator with one current

$$\langle 0|J^a(z)\phi_{\lambda_n}(z_n)\ldots\phi_{\lambda_1}(z_1)|0\rangle = \sum_j \frac{t_j^a}{z-z_j}\langle 0|\phi_{\lambda_n}(z_n)\ldots\phi_{\lambda_1}(z_1)|0\rangle$$

Equivalently we can compute the OPE

$$J^a(w)\,\phi_\lambda(z) = \frac{t^a\phi_{\bar\lambda}(z)}{w-z} + \sum_{l=0}^{\infty}(w-z)^l(J^a\phi_\lambda)^{(-l)}(z) = \sum_{p=0}(w-z)^{p-1}\mathcal{X}^a_{-p}\phi_\lambda(z) \tag{12.8}$$

where the raising operators $\mathcal{X}^a_{-p}$ determine a subset of the descendants in the $\mathfrak{g}$ module of the primary field $\phi_\lambda(z)$; as in the Vir case, the positive modes by definition annihilate the primary, $\mathcal{X}^a_p\phi_\lambda(z) = 0\,, p>0$, as well as do the zero mode elements of the subalgebra $n_+ \ni \mathcal{X}^\alpha_0$. Any of these descendent fields is reproduced by contour integration and in particular the normal product $: J\phi_\lambda := (J\phi_\lambda)^{(0)} = (J\phi_\lambda)$ is computed from

$$\begin{aligned}\int_{C_z}\frac{R(J^a(w)\,\phi_\lambda(z))}{w-z} &= \int_{C_{0,z}}\frac{J^a(w)\,\phi_\lambda(z)}{w-z} - \int_{C_0}\frac{\phi_\lambda(z)J^a(w)}{w-z} \\ &= \sum_{n\geq 1}X^a_{-n}z^{n-1}\,\phi_\lambda(z) + \sum_{n\geq 0}\phi_\lambda(z)z^{-n-1}X^a_n\end{aligned} \tag{12.9}$$

so that $: J^a(w)\,\phi_\lambda : (0)|\lambda\rangle = X^a_{-1}|\lambda\rangle$. The current is not a primary, but a descendant of the identity field and $J^a(0)|0\rangle = X^a_{-1}|0\rangle$.

- Then recall the Sugawara construction: one defines the tensor (9.13), or, in modes

$$L_n = \frac{1}{2(k+h^\vee)}(X_aX^a)_n = \frac{q_{ab}}{2(k+h^\vee)}\big(\sum_{m\geq 1}X^a_{-m}X^b_{n+m} + \big(\sum_{m\geq 0}X^b_{n-m}X^a_m\big) \tag{12.10}$$

and shows that it closes Vir with the Sugawara central charge

$$c = \frac{k\dim \mathrm{g}}{k+h^\vee} \tag{12.11}$$

and that it has an OPE with the currents

$$T(w)J^a(z) \sim \frac{J^a(z)}{(w-z)^2} + \frac{\partial_z J^a(z)}{w-z}$$

This leads to the Sugawara dimension as the eigenvalue of L_0 on the primary fields

$$L_0|\lambda\rangle = \frac{C^{(2)}(X)}{2(k+h^\vee)}|\lambda\rangle = \frac{<\bar\lambda+2\rho,\bar\lambda>}{2(k+h^\vee)}|\lambda\rangle \tag{12.12}$$

• Alternatively, without assuming the Sugawara realisation, the affine algebra can be extended to a semidirect sum with Vir, introducing derivations

$$[L_n, X^a_m] = -m X^a_{n+m}$$

In the module generated by the negative modes of this big algebra there are singular vectors - their factorisation leads to the Sugawara expression for the negative mode Vir generators. E.g., consider

$$(L_{-1} - \gamma\, 2\, q_{ab} X^a_{-1} X^b_0)|\lambda\rangle \tag{12.13}$$

The commutator with L_1 fixes the eigenvalue of $L_0|\lambda\rangle = h|\lambda\rangle$ to be proportional to the value of the second Casimir, while the commutator with some $X_1^{-\alpha_i}$ fixes the constant $1/\gamma = 2(k + h^\vee)$. E.g. in $\widehat{sl}(2)$, take X_1^-

$$X_1^- \big(L_{-1} - 2\gamma(X^+_{-1}X^-_0 + \tfrac{1}{2}X^+_0 X^0_0)|\lambda\rangle = \big(X^-_0 - 2\gamma\,(-X^0_0 X^-_0 + kX^-_0 + X^-_0 X^0_0)\big)|\lambda\rangle$$
$$= (1 - 2\gamma\,(k+2))X^-_0 = 0 \Rightarrow 2\gamma = k+2$$

and we recover in agreement with (12.10) the factor 2

$$(L_{-1} - \tfrac{1}{2(k+2)}(X^a X_a)_{-1})|\lambda\rangle = (L_{-1} - \tfrac{1}{(k+2)}(X^a_{-1} X_{0;a})|\lambda\rangle$$

The higher Sugawara mode combinations become singular after the factorisation of the lower ones.

We can write (12.13) equivalently as a relation for the descendants of fields.

In particular we can interpret this as a quantum version of the definition of the classical currents: take (12.4) with an arbitrary constant κ and replace product with normal product

$$: J(z) g(z,\bar z) := \kappa \frac{\partial}{\partial_z} g(z,\bar z)$$

The l.h.s is defined from the OPE of the current with the primary field

$$\begin{aligned} \lim_{w\to z} \big(t_a J^a(w)\, g(z,\bar z) - \frac{t^a t_a}{w-z} g(z,\bar z)\big) &=: Jg : (z,\bar z) \\ &= 2\mathcal{X}^a_{-1}\mathcal{X}_{0;a}\, g(z,\bar z) = (k+h^\vee)\frac{\partial}{\partial_z} g(z,\bar z)\,, \end{aligned} \tag{12.14}$$

hence the quantised coefficient is $\kappa = (k + h^\vee)$. From the singular term the Sugawara dimension is fixed since $t^a t_a$ gives the value of the second Casimir operators $C^{(2)}$ in the fundamental representation $\bar\lambda = \bar\Lambda_1$.

• Thus in the quantum theory the integer k is shifted to $k + h^\vee$ and the field g transforming according the fundamental irrep of $\bar{\mathfrak{g}}$ receives an anomalous dimension described by the Sugawara dimension (12.12) related to the fundamental integrable representation.

12.3. Knizhnik-Zamolodchikov equations

In general factorising the simplest vector (12.13) leads to partial differential equations for the correlators. Inserting it in a correlator we then move the generators to the other fields, now using the transformation (12.7) defined by the commutators (12.7) with the fields. Repeating this for each field and using that the dual vacuum is annihilated by the positive modes, we get a system of partial differential equations for the correlators.

- KZ equations

$$0 = \big((k+h^\vee)\tfrac{\partial}{\partial z_j} - \sum_{\substack{s=1\\ j\neq s}}^{n} \tfrac{\Omega_{js}}{z_j - z_s}\big) G(z_n, \bar\lambda_n; \ldots; z_1, \bar\lambda_1)\,, \quad , \quad j = 1, 2, ..., n\,, \tag{12.15}$$
$$\Omega_{js} = q_{ab} t_j^a\, t_s^b = \Omega_{st}$$

where q_{ab} is the inverse of the Killing-Cartan form q^{ab}.

The integrability condition for this system

$$[\nabla_i, \nabla_j] = 0\,,\ \nabla_j = (k+h^\vee)\partial_j - \sum_{s\neq j} \frac{\Omega_{js}}{z_{js}} \tag{12.16}$$

is checked to be satisfied by using

$$[\Omega_{12}, \Omega_{13} + \Omega_{23}] = f_{abc}\ t_2^a t_3^b t_1^c + f_{abc} t_1^a t_3^b t_2^c = 0$$
$$[\Omega_{ij}, \Omega_{pt}] = 0\,, \text{for all different } i, j, p, t$$

and denoting

$$r^{ij}(z_i - z_j) := \frac{\Omega_{ij}}{z_{ij}}$$

this implies that

$$[r^{12}(z_{12}), r^{13}(z_{13})] + [r^{12}(z_{12}), r^{23}(z_{23})] + [r^{13}(z_{13}), r^{23}(z_{23})] = 0 \tag{12.17}$$

since then

$$[r^{12}(z_{12}), r^{13}(z_{13})] = \frac{1}{z_{23}}(\frac{1}{z_{12}} - \frac{1}{z_{13}})[\Omega_{12}, \Omega_{13}] \tag{12.18}$$

The equality (12.17) is the classical Yang-Baxter equation (CYBE) and the KZ eqn provides an example for a rational solution of the (CYBE).

- In addition to the KZ system of differential equations the correlators satisfy the algebraic equations coming from the decoupling of the purely $\mathfrak{g}$ singular vectors, as well as all the standard Ward identities with respect to the projective Vir subalgebra and the horisontal subalgebra.

Literature: [46], [47], [49]

13. Solutions of the KZ equations - the $sl(2)$ example

We shall consider in detail the simplest $\mathfrak{g} = \widehat{sl}(2)$ case and it will be convenient to work not with tensors but with functions $\phi(x,z)$ of "isospin" variable; this functional realisation is standardly generalised to any algebra. Then (12.7) is replaced by

$$[J_n^a(w), \phi(x,z)] = z^n \, S^a \phi(x,z) \tag{13.1}$$

with $(-S^a)$ - differential operators realising the sl(2) algebra,

$$S^- = -\partial_x \,, \;\; S^0 = -2x\partial_x + 2j \,, \;\; S^+ = x^2\partial_x - 2jx$$

For the integrable representations the fields are polynomials in x - this is a consequence of the fact that the fields satisfy the equations coming from the factorisation of the "horisontal" singular vectors $(X_0^-)^{2j+1}$, i.e., that they span a finite dimensional irrep of $sl(2)$.

- The 3-point function invariant with respect to the two projective $SU(2)$ groups reads

$$< \phi_{j_3}(x_3,z_3)\phi_{j_2}(x_2,z_2)\phi_{j_1}(x_1,z_1) >= C_{j_3 j_2 j_1} \, \frac{x_{32}^{j_{32}^1} \, x_{31}^{j_{31}^2} \, x_{21}^{j_{21}^3}}{z_{32}^{\triangle_{32}^1} \, z_{31}^{\triangle_{31}^2} \, z_{21}^{trian_{21}^3}}$$

Taking derivatives $\prod_i \partial_{x_i}^{j_i - m_i}$, $\sum_i m_i = 0$ one recovers the Clebsch-Gordan coefficients. Here the scaling dimensions $\triangle_i$ are the Sugawara ones.

Ex: Let us find the integrable fusion rules from the factorisation of the two singular vectors $(X^-)^{2j+1}$ and $(X_{-1}^+)^{k+1-2j}$. The 3-point matrix element is

$$W_3(x,z) = \langle j_3 | \phi_{j_2}(x,z) | j_1 \rangle = C_{j_3,j_2,j_1} \, \frac{x^{j_{12}^3}}{z^{\triangle_{12}^3}}$$

Using the functional realisation (13.1) we have already computed from the factorisation of the horizontal singular vector $(X^-)^{2j+1}$

$$\begin{aligned} &\langle j_3 | \phi_{j_2}(x,z) (X_0^-)^{2j_1+1} | j_1 \rangle = 0 \,, \\ &\Rightarrow a(a-1)....(a-2j_1) = 0 \,, \quad a = j_{12}^3 \,, \\ &\Rightarrow j_3 = j_1 + j_2 \,, \; j_1 + j_2 - 1 \,, \ldots , j_1 - j_2 \,, \end{aligned} \tag{13.2}$$

and exchanging the role of j_1 and j_2, the actual lower bound is $|j_1 - j_2|$.

The factorisation of the second singular vector $(X_{-1}^{+})^{k+1-2j_1}$ also can be easily computed using (13.1),

$$\begin{aligned}
&\langle j_3|\phi_{j_2}(x,z)(X_{-1}^{+})^{k+1-2j_1}|j_1\rangle = 0\,,\\
&\Rightarrow (a-2j_2)(a+1-2j_2)....(a+k-2j_1-2j_2) = 0\,, \quad a = j_{12}^{3}\,,\\
&\Rightarrow j_3 = j_1 - j_2\,,\ j_1 - j_2 + 1\,, \dots\,,\ k - j_1 - j_2
\end{aligned} \tag{13.3}$$

Combining (13.2) and (13.3) the actiual upper bound becomes $\min(k-j_1-j_2, j_1+j_2)$ and we obtain the fusion rules - in general, unlike the analogous computation in the Vir case.

NB: Observation:

Let $\triangle_j = \frac{j(j+1)}{k+2}\,, k+2 = 1/t\ \rightarrow$

$$\lim_{x\to z} W_3(x,z) = C_{j_3,j_2,j_1}\, z^{h_{j_3}-h_{j_1}-h_{j_2}}\,,\ h_j = \triangle_j - j$$

i.e., we recover (up to a normalisation) the correlators and the fusion rules in the Vir theory with such t: non-minimal. If we have started from the non-unitary admissible representations of the affine algebra - rational level and isospin we would recover the minimal theory, though the starting fusion rules are more complicated since the second singular vector is no more a simple power of one generator.

- The system of KZ equations in this case reads

$$\begin{aligned}
&\big(\tfrac{\partial}{\partial z_a} - \tfrac{1}{k+2}\sum_{\substack{b=1\\ b\neq a}}^{n} \tfrac{\Omega_{ab}}{z_a - z_b}\big) W^{(n)}(z_1,x_1,J_1;\dots;z_n,x_n,J_n) = 0 \quad a = 1,2,....n=4\,,\\
&\Omega_{ab} = S_a^{+}S_b^{-} + S_a^{-}S_b^{+} + \frac{1}{2}S_a^{0}S_b^{0}\\
&= -x_{ab}^{2}\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b} + 2x_{ab}\left(J_a\frac{\partial}{\partial x_b} - J_b\frac{\partial}{\partial x_a}\right) + 2J_aJ_b\,,
\end{aligned} \tag{13.4}$$

- The state $|J\rangle = \Phi_J(0,0)|0\rangle$ for a primary field $\Phi_J(x,z)$ represents a highest weight state (h.w.s.) of a Verma module with respect to the generators $X_n\,, L_n$. Similarly $\Phi_J(x,z)|0\rangle = e^{z\,L_{-1} - x\,S^{-}}\,|J\rangle$ can be viewed as a Verma module h.w. state with respect to the "curly" generators determined from the OPE expansions with the current J and the Vir tensor T

$$\begin{aligned}
\mathcal{X}_n^a(x,z)\,\Phi_J(x,z)|0\rangle &: = e^{z\,L_{-1}+x\,X_0^{-}}\ X_n^a\,|J\rangle\,,\\
\mathcal{L}_n(z)\,\Phi_J(x,z)|0\rangle &: = e^{z\,L_{-1}+x\,X_0^{-}}\ L_n\,|J\rangle\,.
\end{aligned} \tag{13.5}$$

Then alternatively the KZ equations are rewritten in a form demonstrating its origin from a singular vector

$$\Big(\mathcal{L}_{-1,a} - \frac{1}{k+2}\,(\mathcal{X}^{+}_{-1,a}\,\mathcal{X}^{-}_{0,a} + \frac{1}{2}\,\mathcal{X}^{0}_{0,a}\,\mathcal{X}^{0}_{-1,a})\Big)\,W^{(n)}(z_1,x_1,J_1;\ldots;z_n,x_n,J_n) = 0\,. \quad (13.6)$$

Here for $n \geq 0$

$$\begin{aligned}
\mathcal{X}^{-}_{-n,a} &= -\sum_{b(\neq a)} \frac{S^{-}_{b}}{z^{n}_{ba}}\,, \quad \mathcal{X}^{0}_{-n,a} = -\sum_{b(\neq a)} \frac{S^{0}_{b} + 2x_a\,S^{-}_{b}}{z^{n}_{ba}}\,,\\
\mathcal{X}^{+}_{-n,a} &= -\sum_{b(\neq a)} \frac{S^{+}_{b} - x_a\,S^{0}_{b} - x^2_a\,S^{-}_{b}}{z^{n}_{ba}} \qquad (13.7)\\
\mathcal{L}_{-n,a} &= \sum_{b(\neq a)} \frac{1}{z^{n-1}_{ba}}\Big(\frac{(n-1)\Delta_{J_b}}{z_{ba}} - \frac{\partial}{\partial z_b}\Big)\,.
\end{aligned}$$

- The correlators depend both on z_a and polynomially on x_a; in the limit $x_n, z_n \to \infty$

$$W^{(n)}(x,z) = \sum_{|\mu|=s}\prod_{a}\frac{x_a^{\mu_a}}{\mu_a!}I_{\mu;\Gamma}(z)\,, \quad \mu = (\mu_1,\ldots\mu_{n-1}) \in \mathbb{Z}^{n-1}_{+} \qquad (13.8)$$

and the general structure of the coefficients in the x_i - power series - multiple integrals over some cycle Γ

$$I_\mu = \prod_{i=1}^{s}\int_\Gamma du_i \Phi(z_a,u_i)\phi_\mu(z_a,u_i)$$

with integrand given by the Vir thermal integrand $\Phi(z_a,u_i)$ - i.e., product of powers $(z_a - u_i)^{-2\alpha_a b}$ and $u_{ij}^{2b^2}$, describing the integrand of the correlator of n vertex operators of charges $\alpha_{j_a} = bj_a\,, b^2 = \frac{1}{k+2}$ and $s = \sum_i j_a - j_n$ screening charges. This integrand is now modified by a meromorphic factor $\phi_\mu(z_a,u_i)$. The various bases of contours - same as the underlying Vir theory, i.e., determined from the singular points of the factor $\Phi(z_a,u_i)$; number of integrals determined by the number s of screening charges - they are of one type; charge conservation $\sum_i j_i - j_n = s$. In particular the solution for the 4-point function can be written as a power series of the anharmonic ratio x with coefficients $I_\mu(z)$.

- This structure of the solution is reproduced also by the *Wakimoto representation* - currents and fields realised by free fields - the logarithmic field $\varphi(z)$ and the pair of bosonic ghosts $\beta(z)$ and $\gamma(z)$ of dimension 1 and 0 respectively

$$\gamma(w)\beta(z) \sim \frac{1}{w-z}$$

The affine generators are realised in terms of these fields as

$$e(z) = \beta(z)\,,$$

$$h(z) = 2(\gamma\beta)(z) - \frac{2i}{b}\partial\varphi(z)\,,$$

$$f(z) = -(\beta(\gamma\gamma))(z) - \frac{2i}{b}\partial\varphi(z) - k\partial\gamma(z)$$

The energy-momentum tensor

$$T = - : b\partial\gamma : - : \partial\varphi\partial\varphi : -ib\partial^2\varphi\,, \quad b^2 = \frac{1}{k+2}$$

i.e., the $\varphi-$ dependent part is changed from $\alpha_0 = 1/b - b$ to $-b$. This reproduces the central charge since the bosonic ghost part $T_{\rm gh}$ has central charge 2

$$2 + 1 - 6b^2 = \frac{3k}{k+2}$$

Combining with the functional realisation the fields are realised as

$$\begin{aligned}\psi_j(x,z) = &\sum_m \binom{2j}{j+m} x^{j+m}\psi_j^m(z) = \sum_m \binom{2j}{j+m} x^{j+m}(-\gamma_j)^{j-m}\, V_{\alpha_j}\\ =&(x-\gamma(z))^{2j}V_{\alpha_j}\,,\\ &\text{or, alternatively, the state } \ \psi_j^j(z) = \beta^{2j}V_{\alpha_0-\alpha_j}\end{aligned} \tag{13.9}$$

The Sugawara dimension of the fields is recovered as

$$\triangle_{\alpha_j} = \alpha_j(\alpha_j + b) = j(j+1)b^2 = \alpha_j(\alpha_j - \alpha_0) + 2j$$

The screening charge is an integral of the vertex of (Sugawara) dimension 1

$$\beta(z)V_{-b}(z) = \beta e^{-2ib\varphi}$$

The modifying meromorphic factor ϕ_μ in the integrand is reproduced by the ghost correlator

$$\langle\gamma(z_{n-1})^{j_n+m_n}....\gamma(z_1)^{j_1+m_1}\beta^{2j_n}(z_n)\beta(u_s)....\beta(u_1)\rangle\,, \ \sum_a(j_a+m_a) = s$$

(taken in the limit $\lim_{z_n\to\infty} z_n^{2j_n} < .. >$).

• Another representation of the 4-point correlator is given by a power series in the difference of the anharmonic ratios x and z

$$W^{(4)} = \sum_t B_t (x-z)^t I_t(z) \tag{13.10}$$

the KZ eqn is reduced to an ordinary differential equation for the coefficient integrals with regular singular points $z = 0, 1, \infty$. This is a matrix differential equation involving 3 terms I_t, $I_{t\pm1}$ at each step. Here generically the first member I_0 of the new basis $\{I_t\}$ is that of the underlying Vir theory, while the last I_s ($s = \min(\sum_{a=1} j_a^3 - j_4, 2j_1, ...2j_4)$) coincides with the WZW correlator. This basis allows to reduce the matrix system of KZ equations and show that it is equivalent to the BPZ differential equation for the first member of the basis, the 4-point Vir correlator [50]. For $s = 2j_a$ for a fixed a the BPZ eqn corresponds to the singular vector of the Virasoro Verma module labelled by the h.w. $h_{1,2j_a+1}$.

• The Fateev-Zamolodchikov [51] solution was based on the observation that the KZ equations for the two variables x, z coincides with the BPZ eqn for a **5-point** Vir correlator with a fundamental field $\phi(x)$ at the fifth point x so that the integrand is modified by the product $\prod_i (x-u_i)^2$. The charges (isospins) $\alpha_{j'} = j_{s'} b$ of this 5-point correlator are linearly related to $\{j_s\}$.

Thus there are two connections with the Vir models - one is related to the quantum DS reduction (QHR) to be discussed in the next lecture, the other bears deep relation with Sklyanin method of separation of variables in integrable models; generalised to Liouville theory and the non-compact WZW model in [52].

• These correlators are easily generalised for $k + 2 = p/p'$ in the cases in which the correlators of the underlying minimal Vir theory are the "thermal" once with only one type of screening charges; the extended WZW correlators are those of a subset of the admissible representations with fractional charge and in general fractional isospin. Braiding and fusing are same as in the minimal models; in fact if we exchange both x and z coordinates - the braiding is identical since the Sugawara dimensions are replaced by the dimensions in the Vir theory.

• To construct the general correlators of the admissible representations one needs a second screening charge, which looks formal $\beta^{-(k+2)}(z) V_{1/b}(z)$, but still the correlator can be given sense through the correspondence with a general 5-point minimal theory correlator.

Comments:

The KZ equations play an important role in mathematical physics and appear in many contexts.

Generalisations:

- other solutions of the CYBE (rational, trigonometric and elliptic); the quantisation of the CYBE led to the theory of quantum groups

- KZB (Bernard) equations for WZW conformal models on the torus - additional term in the KZ eqs; Felder *dynamical* CYBE - led to elliptic quantum groups, etc - solutions for higher rank cases [53]

- another important generalisation: The KZ equations have been generalised to q-KZ [54] so that the affine algebra $\mathfrak{g}$ is replaced by the quantum affine algebra $U_q(\mathfrak{g})$. These are now q- difference eqs and they have been applied to (massive) integrable models for the computation of correlation functions and formfactors. Also the free field Wakimoto realisation has been q- deformed in various ways and used in these constructions.

Literature: [51], [55], [56].

14. From WZW models to new conformal models

The WZW models are basic in RCFT as they can be used to construct new conformal models. We shall review two such methods: the coset construction and the quantum DS reduction (quantum Hamiltonian reduction).

14.1. Coset construction

Let us start with an example, related to the Wakimoto free field representation of the $sl(2)$ WZW model.

For $k = 1$ the contribution of the ghosts and the derivative term in T

$$T = - : \beta\partial\gamma : - : \partial\varphi\partial\varphi : - ib\partial^2\varphi\,, \quad b^2 = \frac{1}{k+2}$$

$$2 + 1 - 6b^2 = \frac{3k}{k+2}$$

are compensated since $6b^2 = 2$ so we are left with a $c = 1$ Vir central charge. For $k = 1$ the primary fields are two, $j = 0, 1/2$ and the states of the field with $j = 1/2$ are realised by $V_{b/2}(z) = e^{ib\varphi}(z)$ and $\gamma(z)V_{b/2}(z)$.

On the other hand the dimensions of the exponential field $e^{2ij\varphi}$ are determined by the $c = 1$ canonical part and so are $h_j^{(c=1)} = j^2$. For $j = \pm 1$ these fields have dims 1 and are descendants of the identity which can be identified with the currents $J^{\pm} = e^{\pm 2\pi i\varphi}$. Indeed one checks that together with $J^0(z) = i\partial\varphi(z)$ they close the algebra $\widehat{sl}(2)_k$ at level $k = 1$; the leading singular term in the OPE of the two exponentials is $1/(w-z)^2$.

We thus get a *vertex representation* of the current algebra at level $k = 1$,

$$J^{\pm}(z) = e^{\pm 2\, i\varphi(z)}\,, \quad J^0(z) = i\partial\varphi(z)$$

The question is can this vertex construction of the $k = 1$ affine algebra be extended to higher values of k? The answer is yes, but we need some new fields so that

$$\begin{aligned} J^+(z) &= \sqrt{2k}\, \psi_1(z)\, e^{2\, i\sqrt{1/k}\, \varphi(z)}\,, \\ J^-(z) &= \sqrt{2k}\, \psi_1^+(z)\, e^{-2\, i\sqrt{1/k}\, \varphi(z)}\,, \\ J^0(z) &= i\sqrt{2k}\, \partial\varphi(z) \end{aligned} \tag{14.1}$$

The fields ψ_1, ψ_1^+ have dimensions $1 - 1/k$ and for $k = 2$ these are just free fermions. For $k > 2$ the fields are parafermions - part of an algebra of k fields including the identity, they satisfy a $\mathbb{Z}_k$ parastatistics, whence their name. This algebra - now non-local, with fractional spins (chiral dims) has representations, etc. We shall not go into the details of their description: just note that they appear (together with more fields) in one of the simplest examples of the general coset construction we are going to describe - the example corresponds to the coset theory denoted

$$\frac{\widehat{sl}(2)_k}{\hat{u}(1)}\,, \qquad \text{of central charge } c = \frac{3k}{k+2} - 1$$

• A GKO (Goddard, Kent and Olive) coset model is a quotient of a WZW (affine algebra may be semisimple - a direct sum of affine algebras) over another WZW model. Many properties of the coset theory are derived from those of the WZW ones. This is also a way to produce automatically unitary models starting from unitary (integrable) representations of the affine algebras. The idea is simple:

given an affine subalgebra $\mathfrak{p} \subset \mathfrak{g}$ the generators of $\mathfrak{p}$ are some linear combinations of the generators of $\mathfrak{g}$ and hence its commutator with the two Vir generators are

$$[L_n^{\mathfrak{g}}, X_m^{a;\mathfrak{p}}] = -mX_{n+m}^{a;\mathfrak{p}} = [L_n^{\mathfrak{p}}, X_m^{a;\mathfrak{p}}]$$

i.e., the generators of $\mathfrak{p}$, and hence the Sugawara Vir generators of the subalgebra commute with the difference of Vir generators

$$\begin{gathered} K_n^{\mathfrak{g};\mathfrak{p}} := L_n^{\mathfrak{g}} - L_n^{\mathfrak{p}} \;\Rightarrow\; [K_n^{\mathfrak{g};\mathfrak{p}}, X_m^{a;\mathfrak{p}}] = 0 \Rightarrow \\ [K_n^{\mathfrak{g};\mathfrak{p}}, L_m^{\mathfrak{p}}] = 0, \text{ or, } [L_n^{\mathfrak{g}}, L_m^{\mathfrak{p}}] = [L_n^{\mathfrak{p}}, L_m^{\mathfrak{p}}] \end{gathered} \tag{14.2}$$

from which one obtains that this difference closes a Vir algebra

$$[K_n^{\mathfrak{g};\mathfrak{p}}, K_m^{\mathfrak{g};\mathfrak{p}}] = (n-m)[K_{n+m}^{\mathfrak{g};\mathfrak{p}}, +(c_g^{\mathrm{Su}} - c_p^{\mathrm{Su}})\delta_{m+n,0}\frac{n^3-n}{12} \tag{14.3}$$

This is the Vir algebra of the coset model and it has central charge - the difference of the two Sugawara central charges.

Let us apply this construction to a case in which the initial algebra is the semisimple algebra $\mathfrak{g} \oplus \mathfrak{g}$ while $\mathfrak{p} = \mathfrak{g}$ as well, all three with different levels. A diagonal coset is defined by embedding $\mathfrak{p}$ diagonally in the sum, i.e., the generators are simply a sum of the generators of each copy of $\mathfrak{g}$, and since the generators of the two copies commute, this implies that the level of $\mathfrak{p}$ is just the sum of the two levels. This is denoted

$$\frac{\mathfrak{g}_{k_1} \oplus \mathfrak{g}_{k_2}}{\mathfrak{g}_{k_1+k_2}}$$

An example is provided by the unitary minimal models $t = \frac{k+2}{k+3}$ arising as diagonal cosets of the affine $\widehat{sl}(2)$

$$\frac{\hat{sl}(2)_k \times \hat{sl}(2)_1}{\hat{sl}(2)_{k+1}}$$

reproducing the central charges

$$c^{\text{Vir}}_{p=k+3,p'=k+2} = c_1 + c_k - c_{k+1} = 1 - \frac{6}{pp'}\,, \quad p' = k+2 \geq 3\,, p = k+3$$

Furthermore one can show that the product of representations of $\mathfrak{g} \oplus \mathfrak{g}$ decomposes into a product of representations of $\mathfrak{p}\oplus$Vir and this is done level by level using the decomposition of the product of characters.

• Character decomposition:

$$\chi_{(\bar\lambda,k)}(\zeta,t,\tau)\,\chi_{\bar\mu,1)}(\zeta,t,\tau) = \sum_\gamma \chi^{\text{Vir}}_{(\lambda,\mu,\gamma)}(\tau)\,\chi_{(\bar\gamma,k+1)}(\zeta,t,\tau)$$

where

$$\bar\lambda + \bar\mu - \bar\gamma \in \bar{Q}$$

and the branching function is precisely a Vir character. There are only 2 integrable representations at level 1, here labelled by the weight μ, and the triple label effectively is replaced by the pair $(\bar\lambda, \bar\gamma)$, being of the same, or opposite $2-ality$, depending on the chosen weight at level 1. The range of integrable representations for

$< \bar\lambda + \overline{\rho}, \alpha >= s,\ < \bar\gamma + \overline{\rho}, \alpha >= r$ is

$$1 \leq s \leq k+1 = p'-1\,, 1 \leq r \leq k+2 = p-1\,,$$

and coincides with the minimal series. Furthermore the $\mathbb{Z}_2$ symmetry is preserved since if the identity representation $\bar\mu = 0$ is taken in the l.h.s., then $r = s$ mod 2 - note that then the character labelled by $(p-r, p'-s)$ does not appear in the r.h.s. since $p-r = (p'-s)+1$ mod 2; analogously taking the character at level 1 with $< \mu, \alpha >= 1$ in the l.h.s. allows only one of the two equivalent Vir representations to appear in the r.h.s.

• Some other alternative coset realisations exist for particular minimal models: e.g. the Ising model with central charge $c = 1/2$ can be obtained as a diagonal coset of the affine algebra $\hat{E}_8$, at level 1 of the two copies, and hence level 2 for the embedded subalgebra; use that $\dim E_8 = 248$ and Coxeter number $h = 30$.

• Furthermore S matrix, fusion multiplicities N can be reproduced as well almost as products of the three WZW quantities.

However not everything is recovered: the characters, being branching functions cannot be expressed as some products of WZW characters, the same applies to the minimal model fields and correlators; one cannot recover the singular vectors. In that respect the second method to be discussed seems better suited apart from the correlators - there is a relation, but the WZW ones are more complicated anyway.

14.2. The quantum Drinfeld-Sokolov reduction

In this approach the minimal series (including the unitary subseries) are reproduced starting from a class of non-unitary fractional level and highest weights - the admissible representations of the affine algebra [28]. These irreps also appear as factors of infinitely reducible Verma modules and are algebraically similar to the integrable h.w. representations.

• Let us start from the classical picture.

We consider the functions $e(x), f(x), h(x)$ and the constant κ as coordinates of the dual $\hat{sl}(2)^*$ of the affine algebra $\hat{sl}(2)$. It is a phase space with Poisson brackets

$$\begin{aligned}
&\{e(x), f(y)\} = 2h(x)\delta(x-y) + \kappa\delta'(x-y) \\
&\{h(x), e(y)\} = e(x)\delta(x-y)\,, \quad \{h(x), f(y)\} = -f(x)\delta(x-y) \\
&\{h(x), h(y)\} = \tfrac{\kappa}{2}\delta'(x-y)
\end{aligned} \tag{14.4}$$

(here $2h$ stands for what was h so far.)

The set of these functions can be associated with the linear matrix differential operator

$$\mathcal{L} = -\kappa\frac{d}{dx} + t^+ f(x) + t^- e(x) + t^0 h = -\kappa\frac{d}{dx} + \begin{pmatrix} h(x) & f(x) \\ e(x) & -h(x) \end{pmatrix}$$

Consider the group of upper triangular matrices

$$\Omega = e^{\alpha(x)t^+} = \begin{pmatrix} 1 & \alpha(x) \\ 0 & 1 \end{pmatrix}$$

which acts in this phase space

$$\mathcal{L} \to \Omega^{-1}\mathcal{L}\,\Omega = \kappa\frac{d}{dx} + \begin{pmatrix} h - \alpha\, e & f - \alpha^2 e + 2\alpha\, h - \kappa\,\alpha' \\ e & -h + \alpha\, e \end{pmatrix}$$

The infinitesimal variation of coordinates is Hamiltonian, i.e., can be represented as a Poisson bracket

$$\delta_\alpha \; g(x) = \{g(x), H_\alpha\}\,, \;\; H_\alpha = -\int e(y)\alpha(y)dy$$

The coordinate $e(x)$ is not affected by this action and we can impose the constraint

$$e(x) = 1$$

This is the first step in the Hamiltonian reduction of the phase space with respect to the gauge group, i.e., we fix the Hamiltonian generator. The particular combination of the variables

$$u(x) = f(x) + h^2(x) - \kappa h'(x) - h^2(e-1) \sim f(x) + h^2(x) - \kappa h'(x) \tag{14.5}$$

commutes with this generator on the constraint

$$\{e(x), u(y)\} \sim 0 \tag{14.6}$$

i.e., it is an invariant of the gauge group on the subspace determined by the constraint. Thus $u(x)$ can be chosen as coordinate of the reduced phase space.

• This invariant polynomial (14.5) of the original variables and their derivatives can be obtained by a "gauge fixing tranformation", i.e., we choose $\alpha = h$, which annihilates the diagonal and reproduces the polynomial (14.5). So the gauge transformation reads

$$\Omega = e^{h(x)t^+} = 1 + h(x)t^+ \tag{14.7}$$

this gives a representative of the gauge orbit; in general the coordinates of the reduced phase space are identified with these orbits.

• Now we compute the new Poisson bracket of the reduced phase space; changing $u \rightarrow u' = u/\kappa$ and skipping the prime

$$\{u(x), u(y)\} = (\frac{du}{dx} + 2u\frac{d}{dx} - \tfrac{\kappa}{2}\frac{d^3}{dx^3})\delta(x-y) \tag{14.8}$$

and we get the Poisson bracket version of the Virasoro algebra. In other words, the Hamiltonian reduction of the (dual of the) affine algebra $\widehat{sl}(2)$ produces (the dual of the) Vir algebra.

The reduced space Poisson brackets provide second Hamiltonian structure of a series of integrable eqs; in this case - the KdV equation. which can be wriiten using (14.8) in a Hamiltonian form

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = \{H, u\}_2 \, , H = \int u^2 dx$$

In the space of the solutions there is another Poisson bracket - Heisenberg type

$$\{u(x), u(y)\}_1 = \delta'(x - y)$$

and another Hamiltonian.

• More general such Poisson algebras encountered in the study of integrable evolution equations are obtained by DS reduction. They are classical versions of the polynomial (non-Lie) W- algebras first introduced by Zamolodchikov. W- algebras find applications in integrable systems, 2d critical phenomena, the Hall effect.

How is the DS reduction quantised?

• More generally there are different approaches to the realisation of the quantum W algebras, e.g., realisation in terms of free fields (quantisation of Miura transformation) - generalised Coulomb gas with no explicit relation to the affine algebras.

Another approach exploits the generalisation of the Felder's BRST approach to the affine algebra representations - irreps obtained as cohomology of some screening charges; then the Fock modules of the affine group (Wakimoto) are extended with fermionic ghosts - effectively the two types of ghost compensate each other through some BRST procedure.

Yet another - BRST cohomology of $\mathfrak{g}$ Verma modules, developed for general reductions.

• The example of $\widehat{sl}(2)$:

First the affine algebra representation space is extended including a pair of fermionic ghost $\mathbf{b}, \mathbf{c}$ of dimensions $0, 1$

$$\mathbf{b}(w)\, \mathbf{c}(z) \sim \frac{1}{w - z}$$

One considers a tensor product of a Verma module of $\mathfrak{g}$ times a Fock module of the ghost system. It is assumed that the vacuum state of any $\mathfrak{g}$ representation is also a vacuum vector for the $\mathbf{b}, \mathbf{c}$ system, i.e.

$$\mathbf{b}_m\, V_0 = \mathbf{c}_n\, V_0 = 0 \;\text{ for }\; m > 0, n \geq 0\, .$$

The constraint $X^+(z) = 1$ is imposed introducing a nilpotent BRST charge

$$Q = \oint_{C_0} dz\; (\mathbf{c}X^+ - \mathbf{c}\,)(z) = (\mathbf{c}\, X^+)_{-1} - \mathbf{c}_0\, , \quad Q^2 = 0$$

the cohomology of Q, KerQ/Im Q in the extended module determines the reduced module.

The Sugawara energy momentum tensor $T^{(\mathrm{sug})} = (X^a X_a)/2(k+2)$ is extended to a field $T^{(\mathrm{tot})}$ commuting with Q,

$$T^{(\mathrm{tot})} = T^{(\mathrm{sug})}(z) + \dot{X}^0(z) + (\dot{\mathbf{b}}\,\mathbf{c})(z)\,. \tag{4}$$

The affine generators are modified by the ghosts, thus the Heisenberg field is shifted

$$\hat{X}^0 = X^0 + (\mathbf{b}\,\mathbf{c})$$

which produces a shift in the central charge $k \to k+2$ in its commutators while the commutators with $X^\pm$ are not changed. Then the total T can be written as

$$\begin{aligned} T^{(\mathrm{tot})} &= T + \frac{1}{k+2}\{Q, (bX^-)\}\,, \\ T &= \frac{1}{k+2}X^- + T^{(\mathrm{f\,f})} = \frac{1}{k+2}\big(X^- + (\hat{X}^0\hat{X}^0) + (k+1)\partial\hat{X}^0\big) \end{aligned} \tag{14.9}$$

inverting and writing in modes

$$\to \; X^-_{-n+1} = (k+2)(L_{-n} - L^{(\mathrm{f\,f})}_{-n}) \tag{14.10}$$

The tensor T, which also commutes with the BRST charge, is the quantum version of the classical polynomial (14.5); the normal product and the use of the shifted with the ghosts Heisenberg currents $\hat{X}^0$ have the effect of shifting the level k in the initial polynomial and changing the central charge. Indeed any of the three tensors $T^{(\mathrm{tot})}$, $T^{(\mathrm{f\,f})}$, and T closes a Virasoro algebra with one and the same value of the central charge

$$c = 13 - 6(k+2+\tfrac{1}{k+2})$$

The tensor $T^{(\mathrm{f\,f})}$ corresponds to the free field realisation of Vir in terms of a current (5.4) $J = \sqrt{\frac{2}{k+2}}\hat{X}^0$.

• To recover the Vir Verma modules or the minimal representations one has to start within the class of fractional level and isospin representations of the affine algebra parametrised by the same pairs of integers and a ratio $k+2 = p/p'$. The correspondence is $2 \to 1$, The representations are characterised by a rational isospin

$$2J_{r,r'} = r - r'\kappa = 2j - 2j'\,\kappa\,,$$

where r, r' are nonnegative integers, restricted furthermore to

$$0 \leq r \leq p-2, \quad 0 \leq r' \leq p'-1\,.$$

Let $r' \neq p'-1$. Upon quantum hamiltonian reduction the representations labelled by $J = J_{r,r'}$ and

$$J^{(1)} = \kappa - J - 1\,, \quad J^{(1)}_{r,r'} = J_{p-r-2,p'-r'-2}\,,$$

both reproduce the Virasoro irreducible representations characterised by the above central charge and conformal weight $h_J = \Delta^{\rm Su}_J - J = h_{J^{(1)}}$,

- The QHR can be used to reproduce the Vir singular vectors from the singular vectors of the affine algebra up to a Q - exact term - for the latter singular vectors there are explicit expressions; in general it produces singular vectors of W algebras and some of them were found precisely by this method.
- The idea is to find a quantum analog of the classical gauge fixing transformation (14.7). Analogously to its classical analog, it effectively converts the affine generators into Vir (or W) algebra) ones, but now it is done for an arbitrary representation, not just the algebra; the effect of this transformation is to remove in (14.9) the dependence on the Heisenberg (hatted) part .

Symbolically

$$R = {}^{\circ}_{\circ} e^{-\sum_{n=1} \frac{\hat{h}_{-n}}{n}(-e)^n} {}^{\circ}_{\circ}$$

meaning a power series with the powers of $e = X_0^+$ put to the right, or

$$\mathcal{R} = \mathbf{1} + \hat{h}_{-1}\, X_0^+ + \frac{1}{2}\Big(\big(\hat{h}_{-1}\big)^2 - \hat{h}_{-2}\Big)\, (X_0^+)^2 + \dots$$

- Properties of $\mathcal{R}$:
  1. leaves invariant KM singular vectors
  2. maps horizontal vectors into the kernel of Q
  3. intertwines KM and Vir generators

More precisely the last property means that for any vector V annihilated by all positive grade generators as well as by the ghost zero mode c_0

$$\mathcal{R}\, X_0^-\, V = (k+2) \sum_{p=0} L_{-p-1}\, \mathcal{R}\; (-X_0^+)^p\, V$$

In particular the simple root singular vector, a power of the horisontal raising generators, $(X_0^-)^{2j+1}v_0$ is equivalent to the singular vector at level $h_{r,1}+r$ in the Vir Verma module $L(h_{r,1},c_k)$, $r=2j+1\,, t=\frac{1}{k+2}$. E.g., for $2j+1=r=2$,

$$\mathcal{R}\,(X_0^-)^2|j\rangle = \tfrac{1}{t}L_{-1}\,\mathcal{R}\,X_0^-|j\rangle - \tfrac{1}{t}L_{-2}\,X_0^+\,X_0^-|j\rangle = \tfrac{1}{t^2}\left((L_{-1})^2 - 2tL_{-2}\right)|j\rangle \qquad (14.11)$$

• In the higher rank cases there are increasingly many DS reductions (already classically) besides the principal one, generalising the $sl(2)$ case, which is maximal. They correspond to different choices of the constraints, fixing part of the generators of the subalgebra n_+. The possible choices correspond to the various ways of embedding of $sl(2)$ as a subalgebra into the horisontal algebra $\bar{\mathfrak{g}}$, and such embeddings are classified. In this way Vir is always a subalgebra, i.e., W is an extension of Vir and furthermore the remaining generators are primary fields with respect to this Vir. Thus one gets many W - algebras. For $sl(n)$ there is a sl(2) embeddings for any partition of n. E.g., for $sl(3)$ there are two - in one of them W_3 - survive two gauge invariants - T and the dimension 3 operator W, in the other, $W_3^{(2)}$ - survive 4 invariants, two of them correspond to affine currents.

The quantisation is analogous: $J(z)^a, b^\alpha(z), c_\alpha(z)$ for any positive root α; for each reduction enter as many ghosts as are the constraint generators. The BRST operator takes the general form

$$Q = \oint_{C_0} dz\,\left((J^\alpha(z) - \chi(J^\alpha(z))c_\alpha(z) - \tfrac{1}{2}f^{\alpha\beta}{}_\gamma(b^\gamma(c_\alpha c_\beta))(z)\right)$$

This - on the level of the algebra. However on the level of representations (besides the principal reduction) there are in general still many uncertainties. One can also reduce the affine algebra characters to those of the W - algebras. Yet there are few results beyond the principal reduction.

Literature: [57], [58], [59], [60], [61].

15. CFT in the presence of boundaries, classification of conformal boundary conditions

Origin of some of the ideas - the open string theory: world sheet - a manifold with boundaries. In statistical mechanics - pioner work by Cardy [62], [63].

Up to now - CFT on the plane: two Vir algebras, or two chiral algebras extending Vir, acting on the physical fields $\phi_{h,\bar{h}}(z,\bar{z})$ where $\bar{z}$ - complex conjugate to z.

- Now in the presence of boundaries we should restrict to conformal transformations which preserve these boundaries. The simplest geometry is that of the (upper) half - plane H_+ with the real axis as a boundary. Then the conformal transformations $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(z)$ should be real analytic functions, $\overline{\epsilon(z)} = \epsilon(\bar{z})$, i.e., the coefficients ϵ_n in $\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}$ should be real: this leaves essentially one set of conformal transformations. In terms of the stress tensors

$$T(z) = \bar{T}(\bar{z}) \;\; \text{on the real axis } z = \bar{z}$$

Hence $T_{xy} = i(T(z) - \bar{T}(\bar{z})) = 0$ for Im$z = y = 0$, which is natural to interpet as a requirement that there is no momentum flow across the boundary; the zero mode - momentum $P = x\partial_y - y\partial_x = x\partial_y = 0$ at $y = 0$.

- This implies that $\bar{T}$ can be identified with the analytic continuation to the lower half plane of the upper half plane tensor $T(z)$, defined for Im $z > 0$. Thus we are left with one Vir algebra. For the modes we take a sum of two integrals one in the upper half-plane along a closed contour C_+, which goes anticlockwise and consists of a semicircle in the upper half-plane and a portion of the real line and its mirror image C_- in the lower half-plane, also oriented anti-clockwise.

$$\begin{aligned} &= \frac{1}{2\pi i}\oint_{C_+} dz\, z^{n+1}\, T(z) \,+ \frac{1}{2\pi i}\oint_{C_-} dz\, z^{n+1}\, \bar{T}(z) \,= L_n + \bar{L}_n \\ &= \frac{1}{2\pi i}\oint_{C} dz\, z^{n+1}\;\, T^H(z) = L_n^{(H)} \end{aligned} \tag{15.1}$$

In the second line we have identified $T(z) := \bar{T}(z)$ for Im$z < 0$; C is the complete circle in the full plane, $C = C_+ + C_-$, with the pieces along the real line cancelled due to the gluing condition $T(x) = \bar{T}(x)$; (15.1) can be also written as an integration over the same contour

$$\frac{1}{2\pi i}\oint_{C_+} dz\, z^{n+1}\, T(z) \,- \frac{1}{2\pi i}\oint_{C_+} d\bar{z}\, \bar{z}^{n+1}\, \bar{T}(\bar{z})$$

where $\bar{z} \in H_-$.

• The first line in (15.1) represents $L_n^{(H)}$ as a sum of the initial generators $L_n + \bar{L}_n$. Thus their action on the fields $\phi_{h,\bar{h}}(z,\bar{z})$, where from now on $z \in H_+$ and $\bar{z}$ is the complex conjugation of $z \in H_+$, can be written as (skipping sometimes H in the notation)

$$[L_n^H, \phi_{h,\bar{h}}(z,\bar{z})] = (D_{n;h,z} + D_{n,\bar{h},\bar{z}})\phi_{h,\bar{h}}(z,\bar{z}) \tag{15.2}$$

In particular, L_{-1}^H acts as $\partial_x = \partial_z + \partial_{\bar{z}}$, etc. From this transformation law we can compute the correlator

$$\langle 0|T^H(z)\phi_{h_n,\bar{h}_n}(z_n,\bar{z}_n)\ldots\phi_{h_1,\bar{h}_1}(z_1,z_1)|0\rangle =$$
$$\sum_j \Big(\frac{h_j}{(z-z_j)^2} + \frac{1}{(z-z_j)}\partial_{z_j} + \frac{\bar{h}_j}{(z-\bar{z}_j)^2} + \frac{1}{(z-\bar{z}_j)}\partial_{\bar{z}_j}\Big)\langle 0|\phi_{h_n,\bar{h}_n}(z_n,\bar{z}_n)\ldots\phi_{h_1,\bar{h}_1}(z_1,\bar{z}_1)|0\rangle \tag{15.3}$$

defined for any z in the full plane. For $z \in H_+$ the combination $\hat{T}(z,\bar{z}) = T^H(z) - T^H(\bar{z}) = \sum_n L_{-n}^H(z^{n-2} - \bar{z}^{n-2})$ represents the half - plane descendant of the identity.

• The transformation rule (15.2) restricted to the three projective generators implies that the 1-point functions of the half-plane scalar fields are non-vanishing,

$$\langle 0|\phi_{h,\bar{h}}(z,\bar{z})|0\rangle = \frac{C_h}{(z-\bar{z})^{2h}}\delta_{h,\bar{h}} = \frac{C_h}{(2y)^{2h}}\delta_{h,\bar{h}}\,, \quad \mathrm{Im}\ z > 0 \tag{15.4}$$

Compare with the full plane fields: the 1-point function there is required to be invariant also with respect to $i(L_{-1} - \bar{L}_{-1}) = \partial_y$ - because of the additional constraint the only correlator annihilated both by ∂_x and by ∂_y is the one of the identity field, i.e. a constant.

The reason behind this invariance is that the action of the generators (15.2) is like the (coproduct) action on a 2-point chiral correlator and indeed (15.4) is a composition of chiral fields located at complex conjugated coordinates. It is then obvious that the 2-point function of the half-plane bulk fields $\phi(z,\bar{z})$ is not fixed by the invariance with respect to the projective subgroup of (15.2), since e.g. the ratio

$$z = \frac{(z_1-z_2)(\bar{z}_1-\bar{z}_2)}{(z_1-\bar{z}_1)(z_2-\bar{z}_2)} = \frac{x_{12}^2 + y_{12}^2}{4y_1y_2}\,,$$

is obviously translation invariant along the real axis, etc. In general the n-point half-plane *bulk* correlator will have the structure of some linear combination of 2n-point chiral blocks.

• This can be extended to theories with additional symmetry, like the WZW models. Unlike the Virasoro symmetry, there is still some freedom in the "gluing" of the right and left currents - this can be done up to some automorphism Ω so that on the real line $J = \Omega(\bar{J})$.

In such a theory we should also expect that the fields on the boundary are "physical", i.e., some chiral fields can be interpreted as physical. The conformal transformations on the real line are generated by the real Vir algebra, in particular the 3-dimensional projective algebra is the real form $sl(2, \mathbb{R})$.

15.1. Boundary conditions and boundary fields - heuristic picture

On an infinitely long strip of width L the Hamiltonian - a generator of translations along the strip $[\partial_t = z\partial_z + \bar{z}\partial_{\bar{z}}]$ is given by the shifted mode (15.1)

$$H = \frac{\pi}{L}(L_0^{(H)} - \frac{c}{24})$$

computed as in (4.5) for $z = e^{\frac{\pi}{L}w}$ with a factor 2 less.

$$T^{\rm strip}(w) = (\frac{\pi}{L})^2 \big(T^H(z) z^2 - \frac{c}{24}\big)\,,\ z = e^{\frac{\pi}{L}w} = e^{\frac{i\pi v}{L}} e^{\frac{\pi t}{L}}\,, 0 \le v \le L\,, 0 \le t \le \infty$$

It is assumed that the boundary conditions are the same on both sides of the strip (resp., on the positive or negative real axis). But we can assume instead that there are different boundary conditions on the two sides, hence the Hamiltonian depends on them, so we assign some labels H_{ba}. The eigenstates may not belong to one irreducible representation, so we shall denote by ${n_{ja}}^b$ the multiplicity with which the representation j appears, or

$$\mathcal{H}_{ba} = \oplus_{i\in\mathcal{I}}\, {n_{ia}}^b \mathcal{V}_i\,. \qquad (15.5)$$

• Looked at the half-plane, the effect of different boundary conditions implies that the translation invariance along the real axis is violated, i.e., the "vacuum" is not annihilated by L_{-1}. This can be interpreted as a presence of a field at $x = 0$ where the change of the boindary conditions happens: ${}^b\phi_j^a(0)|0\rangle$; if it transforms as a representation labelled by j, then the multiplicity ${n_{ja}}^b$ can be also interpreted as the multiplicity of such fields ${}^b\phi_j^a$. The identity field does not change the boundary conditions, so ${n_{1a}}^b = \delta_{ab}$.

Main conclusion: insertion of different boundary conditions - equivalent to insertion of boundary fields. In some sense these fields intertwine boundaries, so initial and final indices are correlated, so e.g., in the 2-point function

$$\langle 0|{}^a\phi_j^b(x_2)\,{}^b\phi_{j^*}^a(x_1)|0\rangle = C_{a\,,j\,,b}\ \frac{1}{|x_{21}|^{2h_j}}\,, \qquad x_2 \ne x_1$$

Similarly 3-point.

• Comment: It is not a coincidence that we use the same notation as for the matrix representations of the Verlinde algebra: we shall prove this.

15.2. CFT on an annulus

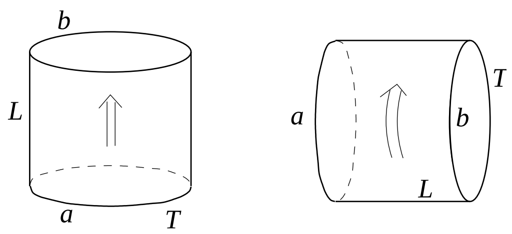
Consider now a strip with periodic boundary condition along the time direction, $w \sim w+T$. i.e., a cylinder; in the half plane $z \sim z e^{\frac{\pi T}{L}}$ i.e. the half circles in H_+ are identified.

The partition function $Z_{b|a}$ on a finite cylinder $T \times L$ with boundaries on its two ends labelled by a, b is now linear in the characters since the space of states (15.5) decomposes into representations of the only chiral algebra

$$Z_{b|a} = \mathrm{Tr}_{\mathcal{H}_{ba}} e^{-TH_{ba}} = \mathrm{Tr}_{\mathcal{H}_{ba}} q^{L_0 - \frac{c}{24}} = \sum_{i \in \mathcal{I}} n_{ia}{}^{b} \chi_i(\tau) \ , \qquad q = e^{-\pi \frac{T}{L}} = e^{2\pi i \tau} \tag{15.6}$$

here $\tau = i\frac{T}{2L}$ is pure imaginary.

- In (15.6) the partition function $Z_{b|a}$ represents a periodic time evolution $e^{-TH_{ba}}$, on the cylinder, whence the trace, with boundary conditions a, b.
- We can also compute the same partition function in a different alternative way,



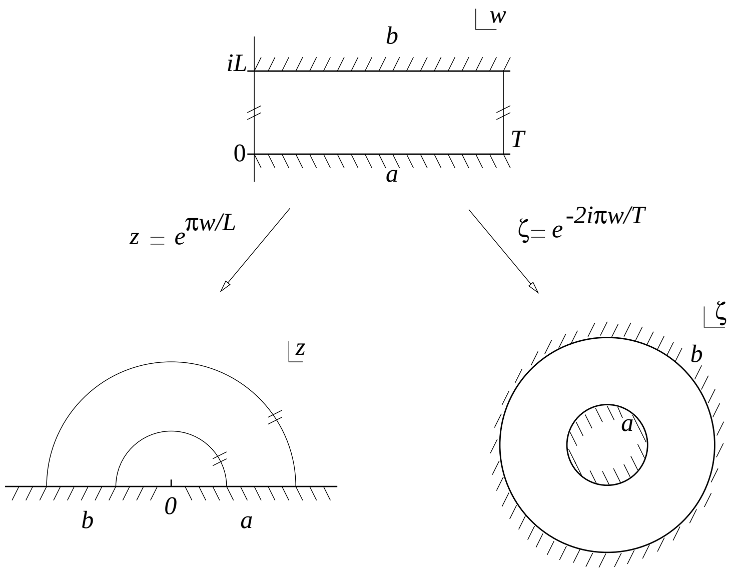
(a) (b)

namely, as a matrix element of the evolution operator $e^{-L\, H^{(cyl)}}$ between some boundary states $|a\rangle$ and $\langle b|$, yet to be determined. Indeed mapping the cylinder (now with periodic "time") to an **annulus domain in the full plane**, $\zeta = e^{-\frac{2\pi i}{T} w}$,
we can express this "space" generator $\mathcal{H}^{(cyl)}$ $[\partial_v = \frac{2\pi}{T}(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}})]$
in terms of the sum of the zero modes of the two Vir algebras

$$H^{(cyl)} = \frac{2\pi}{T}(L_0 + \bar{L}_0 - \frac{c}{12}) \tag{15.7}$$

This computation is once again analogous to the one in (4.5), only "time" is now "space".

Thus the two maps $w \to \zeta$ and $w \to z$ convert the generators of translations along the two directions of the cylinder into zero Vir modes on the plane, or the half- plane respectively.



The same domain seen in different coordinates: a semi-circular annulus, with the two half-circles identified, a rectangular domain with two opposite sides identified, and a circular annulus.

- We obtain for the partition function

$$Z_{b|a} = \langle b|e^{-LH^{\rm cyl}}|a\rangle = \langle b|(\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|a\rangle\,, \qquad \tilde{q}^{\frac{1}{2}} = e^{-\frac{\pi i}{\tau}} = e^{\pi i\tilde{\tau}} \tag{15.8}$$

- The two expressions (15.6) and (15.8) for the same partition function are related by a modular transformation for $\tau = iT/2L$. We can expect that their comparison gives a consistency condition on the integers ${n_{ia}}^b$, analogous to the constraint on the multiplicity Z_{ij} of spin fields implied by the modular invariance on the torus. In the string theory interpretation the two alternative ways of computation of the partition function are referred to as the open string and closed string sectors: "tree level closed string amplitude" versus "one loop open string amplitude".

- How to compute (15.8)? It is computed in the full physical space (11.1) but we need to describe the boundary states $|a\rangle$. They satisfy the gluing (continuity) condition on the two Vir algebras which reads for the boundaries of the annulus domain of the plane

$$\zeta^2 T(\zeta) = \bar{\zeta}^2 \bar{T}(\bar{\zeta}) \quad \text{for } |\zeta|^2 = 1\,, \text{ and } |\zeta|^2 = e^{2\pi L/T}$$

and can be formulated as a condition on the boundary state associated with the inner boundary: it is equivalent to $\zeta^2 T(\zeta) = \zeta^{-2}\bar{T}(\zeta^{-1})_{|\zeta|=1}$, or, in modes

$$(L_n - \bar{L}_{-n})|a\rangle \ = 0\,. \tag{15.9}$$

Similar conditions hold for the generators J of the extended algebra, with a relative sign depending on their spin, and eventually twisting with some automorphism one of the generators.

• The equation can be solved in each of the representations $\mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}}$ appearing in the full space (11.1) [64]. A possible argument makes a formal use of the Schur lemma:

To any state $A = \sum_{N,\bar{N}} a_{N,\bar{N}} \, |j,N\rangle \otimes |j,\bar{N}\rangle \in \mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}}$, where $|j,N\rangle$ are normalised states, $\langle j',N'|j,N\rangle = \delta_{j,j'}\delta_{NN'}$, there corresponds a homomorphism $X_A = \sum_{N,\bar{N}} a_{N,\bar{N}} \, |j,N\rangle \otimes \langle j,\bar{N}| : \mathcal{V}_j \to \mathcal{V}_{\bar{j}}$. The condition (15.9) for the state A, using also the Hermitian conjugation of the Vir generators $L_{-n} = L_n^+$

$$\begin{aligned} 0 &= (L_n \otimes I - I \otimes L_{-n})A \\ &= \sum_{N,\bar{N}} a_{N,\bar{N}} \, L_n|j,N\rangle \otimes |j,\bar{N}\rangle - \sum_{N,\bar{N}} a_{N,\bar{N}} \, |j,N\rangle \otimes L_n^+|j,\bar{N}\rangle \\ \text{implies} \ \Rightarrow \qquad & L_n \, X_A = X_A \, L_n \end{aligned} \tag{15.10}$$

i.e., X_A intertwines the action of the generators in the two representation spaces $\mathcal{V}_j$ and $\bar{\mathcal{V}}_{\bar{j}}$. Since the representations are irreducible, they must then be equivalent, which means that $j = \bar{j}$. Hence the solutions of (15.9) are restricted to states in the diagonal products $\mathcal{V}_j \otimes \bar{\mathcal{V}}_j$, while the intertwiners should be proportional to projectors; the normalisation can be fixed so that the states A are given as

$$|j\rangle\rangle = \sum_N |j,N\rangle \otimes |j,N\rangle \tag{15.11}$$

and X_A are then projectors P_j; these are the Ishibashi states.

Since the trace is computed in the physical space (11.1), we have to take Z_{jj} copies of the space $\mathcal{V}_j \otimes \mathcal{V}_j$, and hence each Ishibashi state should appear with the multiplicity Z_{jj}; $|j,a\rangle\rangle$, $\alpha = 1, 2, \ldots Z_{jj}$.

• Thus finally equation (15.9) determines the boundary states $|a\rangle$ as a linear combination of canonical "Ishibashi" states $|j\rangle\rangle$

$$|a\rangle = \sum_{j,\alpha=1,\cdots,Z_{jj}} \frac{\psi_a^{(j,\alpha)}}{\sqrt{S_{1j}}} |j,\alpha\rangle\rangle \, . \tag{15.12}$$

The summation in (15.12) runs over the subset of $\mathcal{I}$ denoting the scalars (j,j), taken with multiplicity, i.e., over the exponents set $\mathcal{E}$.

• On the Ishibashi states one computes the trace in (15.8) taking into account that on the diagonal of the two representation spaces the eigenvalues of the two L_0 modes are identical so they simply add, effectively removing the 1/2 factor in $\tilde{g}$, and one obtains

$$\langle\langle k,\alpha|(\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|j,\beta\rangle\rangle = \delta_{kj}\delta_{\alpha\beta}\chi_j(-1/\tau) \tag{15.13}$$

The boundary states are labelled by some set $\mathcal{V} \ni a$ with an involution $a \to a^*$ - so that one can identify the conjugated state $\langle a|$ by a coefficient $\psi^l_{a^*}$ and we shall assume that the coefficients in (15.12) satisfy

$$\psi^\mu_{a^*} = \psi^{\mu^*}_a = (\psi^\mu_a)^*\,. \tag{15.14}$$

The dual state is defined as

$$\langle b| := \sum_{j\in\mathcal{E}}\langle\langle j|\frac{\psi^j_{b^*}}{\sqrt{S_{1j}}}$$

The result for the partition function (15.8) computed on an annulus domain of the plane is then

$$Z_{b|a} = \sum_{j\in\mathcal{E}}\frac{\psi^j_a\psi^{j\,*}_b}{S_{1j}}\chi_j(\tilde{q}) = \sum_{i\in\mathcal{I}}\big(\sum_{j\in\mathcal{E}}\frac{\psi^j_a\psi^{j\,*}_b}{S_{1j}}S_{ji}\big)\chi_i(q) \tag{15.15}$$

• Performing a modular S transformation $-1/\tau \to \tau$ on the characters as in the second equality in (15.15) one compares the coefficients in front of the characters $\chi_i(q)$ in the two alternative expressions (15.15) and (15.6) for the partition function.
• More precisely to extract a relation the set of characters should be linearly independent. In the affine algebra case one has to work with the full, non-specialised characters, so we have to extend the consideration to a Hamiltonian which includes the Cartan generators - which is possible to be done.

• This consistency condition resulting from the comparison of the two alternative expressions for the cylinder partition function is known as **"Cardy equation"**. Originally it was formulated and solved only in the diagonal case, referring to the modular invariants. In that case the set of exponents $\mathcal{E}$ describing the scalars, coincides with the representation set $\mathcal{I}$, so that the sum over j in (15.15) runs over the full set $\mathcal{I}$ and one derives that the coefficients in (15.12) are given by the modular matrix, thus recovering the Verlinde formula. In general we obtain the formula

$$n_{ia}{}^b = \sum_{(j,\alpha)\in\mathcal{E}}\frac{S_{ij}}{S_{\mathbf{1}j}}\,\psi^{(j,\alpha)}_a\,\psi^{(j,\alpha)\,*}_b\,, \qquad a,b\in\mathcal{V}\,,\quad i\in\mathcal{I}\,, \tag{15.16}$$

as a consequence of the relations (15.14) and analogous ones for the S matrix we have

$$n_{ja}{}^{b} = n_{jb^*}{}^{a^*} = n_{j^*b}{}^{a} \tag{15.17}$$

Furthermore we may assume that (15.12) defines a unitary change of basis. In particular imposing the first of the relations

$$\sum_{\alpha\in\mathcal{E}} \psi_a^\alpha \psi_b^{\alpha\,*} = \delta_{ab}\,, \; \sum_{a\in\mathcal{V}} \psi_a^\alpha \psi_a^{\beta\,*} = \delta_{\alpha\beta}\,, \tag{15.18}$$

is equivalent to the orthonormality of the boundary states $|a\rangle$. It is consistent with the inner product which one can define for the Ishibashi states using the asymptotics for $\tilde{q} \to 1$ in (15.13); the factor S_{1j} cancels the one in the normalisation of the boundary states (15.12); in the non-unitary cases one adapts the definition of the norm to get the same final result. The second equality in (15.18) is a requirement of completeness of the set of boundary states. This in particular implies that the cardinality of the two sets is the same, $|\mathcal{V}| = |\mathcal{E}|$.

• The formula (15.16) then states that the multiplicity matrix n_i is diagonalised by the boundary state coefficients ψ_a^j. Using the fact that the ratios of S matrix elements represent the Verlinde algebra

$$\frac{S_{il}}{S_{1l}}\frac{S_{jl}}{S_{1l}} = \sum_{k\in\mathcal{I}} \mathcal{N}_{ij}{}^{k}\,\frac{S_{kl}}{S_{1l}} \tag{15.19}$$

and the completeness condition (the second equality in (15.18)), we come to the conclusion that the multiplicities in (15.5) satisfy the Verlinde fusion algebra, the NIM-reps equation (11.7) which we have already discussed; (15.16) is identical to (11.6)

$$n_i n_j = \sum_s \mathcal{N}_{ij}{}^{s} n_s \tag{15.20}$$

The orthonormality (15.18) implies that the matrix $n_1 = I$, i.e., is the unit matrix and from (15.17) $n_{i*} = n_i^T$, i.e., the set of these commuting matrices contains their transposed, thus they are normal. Vice versa the matrices representating of the fusion algebra with these properties can be simultaneously diagonalised in an orthonormal basis.

• Thus the classification of a complete set of conformal boundary conditions is reduced to the classification of the NIM-reps, the non-negative integer valued matrix representations of the Verlinde fusion algebra. In turn there are naturally associated with graphs, the vertices of which thus parametrise the conformally invariant boundary conditions. Typically it is sufficient to consider the graphs with adjacency matrices $n_{\bar{\Lambda}_i}$ labeled by the fundamental representations, which generate the whole fusion ring.

• The formulation of the problem in terms of the NIM-reps of the Verlinde fusion algebra effectively reduces it in the $sl(2)$ related cases to a problem already studied in [36].

• Namely it reduces to the known classification of the symmetric, irreducible, non-negative integer valued matrices of spectrum $|\gamma_j| < 2$ ($\gamma_j = S_{2j}/S_{1j} = 2\cos\frac{\pi j}{k+2}$; these are the eigenvalues of the adjacency matrix n_2. The classification enumerates these matrices as the adjacency matrices of the graphs $A - D - E - T$, i.e., the classification effectively parallels the ADE classification of the corresponding $\widehat{sl}(2)$ modular invariants, up to one more solution - the tadpole graph $T = A_{2n}/\mathbb{Z}_2$. This graph has to be discarded since its spectrum, i.e., the set $\mathcal{E}$ in this case does not match any of the spectra of the $\widehat{sl}(2)$ modular invariants.

• The immediate consequence is the classification of the conformal boundary conditions for the integrable sl(2) WZW models. Thus the boundary conditions are parametrised by the nodes of the ADE Dynkin diagrams and the required data, like the coefficients ψ_a^j in (15.12), or the NIM-reps themselves, are explicitly known.

• The classification for the minimal $c < 1$ Vir models again parallels that of the modular invariants of those models.

Literature: [62], [63], [65]

16. Boundary CFT: string examples; boundary fields

16.1. A string example

The 1d closed bosonic string moving in time is described by a free field

$X(x^+, x^-) = X_L(x^+) + X_R(x^-)$, where $x^\pm = t \pm \sigma$ are the Minkowski space-time light cone coordinates - analogs to the strip coordinates $w, \bar{w}$.

In Euclidean space and ζ - annulus coordinates, $\zeta = e^{-\frac{2\pi i}{T} w} = e^{\frac{2\pi v}{T} - \frac{2\pi i u}{T}}$; u variable was periodic, had the meaning of time. Here we will employ the inverse interpretation, $-2\pi u/T \to \sigma$, interpreted as "space", periodic with period 2π, while $\frac{2\pi v}{T}$ with $0 \leq v \leq L$ - now has the meaning of euclidean time. In Minkowski space it becomes imaginary, ζ goes to $e^{i(t+\sigma)} = e^{ix^+}$ where $0 \leq t \leq \pi/\tau$; then $\bar{\zeta}$ goes in Minkowski space to $e^{i(t-\sigma)} = e^{ix^-}$. The boundaries are at $t = 0$ and $t = \pi/\tau$.

$$
\begin{aligned}
X_L(x^+) &= \tfrac{1}{2}\big(\hat{x} + (\tfrac{\hat{p}}{2} + r\hat{\omega})x^+ + i\sum_{n\neq 0} \tfrac{a_n}{n} e^{-inx^+}\big) \\
X_R(x^+) &= \tfrac{1}{2}\big(\hat{x} + (\tfrac{\hat{p}}{2} - r\hat{\omega})x^- + i\sum_{n\neq 0} \tfrac{\bar{a}_n}{n} e^{-inx^-}\big)
\end{aligned}
\tag{16.1}
$$

The field $X = X_L + X_R$ is compactified to take values on a circle of radius r imposing the shifted periodicity condition

$$
X(t, \sigma + 2\pi) = X(t, \sigma) + 2\pi r\hat{\omega} \tag{16.2}
$$

$\hat{\omega}$ - integer eigenvalues, winding number. Commutation relations

$$
[a_n, a_m] = n\delta_{n+m,0}\ ,\ \ [\bar{a}_n, \bar{a}_m] = n\delta_{n+m,0}\ ,\ \ [\hat{x}, \hat{p}] = i
$$

where center of mass momentum $\hat{p}$ and coordinate $\hat{x}$ of the string and the winding number operator $\hat{\omega}$ are self-adjoint, while $a_n^+ = a_{-n}\ ,\ \ \bar{a}_n^+ = \bar{a}_{-n}$. Denote also

$a_0 = \frac{\hat{p}}{2} + r\hat{\omega}\, , \bar{a}_0 = \frac{\hat{p}}{2} - r\hat{\omega}$.

The currents of the $\hat{u}(1)$ algebra:

$J_L(x^+) = 2\partial_{x^+} X_L(x^+) = \sum_{n\in\mathbb{Z}} a_n e^{-inx^+}\ ,\ J_R(x^-) = 2\partial_{x^-} X_L(x^-) = \sum_{n\in\mathbb{Z}} \bar{a}_n e^{-inx^-}$

The h.w. irreps of h.w. are generated by the negative modes, with states that satisfy

$$
a_0|\alpha\rangle \otimes |\bar{\alpha}\rangle = \alpha|\alpha\rangle \otimes |\bar{\alpha}\rangle\ ,\ \ \bar{a}_0|\alpha\rangle \otimes |\bar{\alpha}\rangle = \bar{\alpha}|\alpha\rangle \otimes |\bar{\alpha}\rangle
$$

also denoted by the eigenvalues k/r of $\hat{p}$ and rw of $\hat{\omega}$; since X is compact the spectrum of $\hat{p}r$ and $\hat{\omega}$ is given by integers,

$$\alpha = \frac{1}{\sqrt{2}}(kb + \omega/b)\,, \bar{\alpha} = \frac{1}{\sqrt{2}}(kb - \omega/b)\,,\ b^2 = \tfrac{1}{2r^2}$$

notation for the states $|\alpha\rangle \otimes |\bar{\alpha}\rangle = |(\omega, k)\rangle$, dependence on the descendants index suppressed

- Impose **Dirichlet** boundary condition:

$$\begin{aligned} \partial_\sigma X(t,\sigma)|B\rangle_D = 0 \quad \text{for } t = 0, \pi/\tau \\ \Rightarrow (a_n - \bar{a}_{-n})|B\rangle_D = 0\,, \quad n \in \mathbb{Z} \\ \Rightarrow (J_L(x^+) - J_R(x^-))_{t=0}|B\rangle_D = 0 \end{aligned} \tag{16.3}$$

Solved by the coherent ("Ishibashi") state with $\bar{\alpha} = \alpha$, i.e., $w = 0$

$$|(0,k)\rangle\rangle_D = \exp\big(\sum_{n=1}^{\infty} \tfrac{1}{n} a_{-n}\bar{a}_{-n}\big)|(0,k)\rangle_D \tag{16.4}$$

recall $|(0,k)\rangle = |\alpha\rangle \otimes |\alpha\rangle$. A linear combination of such states

$$|B(x_a)\rangle_D = \tfrac{1}{\sqrt{2r}} \sum_{k \in \mathbb{Z}} e^{-ikx_a/r} |(0,k)\rangle\rangle_D \sim \delta(\hat{x} - x_a)\,|(0,k)\rangle_D \tag{16.5}$$

satisfies the stronger condition

$$X(0,\sigma)|B(x_a)\rangle_D = x_a\,|B(x_a)\rangle_D$$

In the last expression in (16.5) - string coordinate takes a fixed value x_a along the boundary where the state $|B(x_a)\rangle_D$ is placed. Note the analogy with $\psi^k_{x_a}$ in (16.5).

Thus the Diriclet closed string is given by

$$X^{(D)} = X_L + X_R = \tfrac{1}{2}\big(x_a + \hat{p}t + i \sum_{n \neq 0} \tfrac{a_n}{n}(e^{-inx^+} - e^{inx^-})\big)$$

- Impose **Neumann** boundary condition:

$$\begin{aligned} \partial_t X(t,\sigma)|B\rangle_D = 0 \quad \text{for } t = 0, \pi/\tau \\ \Rightarrow (a_n + \bar{a}_{-n})|B\rangle_D = 0 \\ \Rightarrow (J(x^+) + \bar{J}(x^-))_{t=0}|B\rangle_D = 0 \end{aligned} \tag{16.6}$$

Solved by the coherent state

$$|(\omega,0)\rangle\rangle_N = \exp\big(-\sum_{n=1}^{\infty}\tfrac{1}{n}a_{-n}\bar a_{-n}\big)|(\omega,0)\rangle_N \tag{16.7}$$

where $|(\omega,0)\rangle = |\alpha\rangle\otimes|-\alpha\rangle$. Now

$$X^{(N)} = X_L + X_R = \tfrac{1}{2}\big(\hat x + 2r\,\hat w\,\sigma + i\sum_{n\neq 0}\tfrac{a_n}{n}(e^{-inx^+}+e^{inx^-})\big)$$

NB: Neumann b.c. would not be possible for a non-compactified closed string.

The general solution is a linear combination of the states (16.7),

$$|B(\tilde x_a)\rangle_N = \sqrt{r}\sum_{w\in\mathbb{Z}} e^{-2ir\omega\tilde x_a}|(\omega,0)\rangle\rangle_N \tag{16.8}$$

- In both cases the Vir generators satisfy the same relation

$$(L_n-\bar L_{-n})|B(x)\rangle_D = 0 = (L_n-\bar L_{-n})|B(\tilde x)\rangle_N$$

- Generalisation to a d-torus, X^μ, describing Dirichlet p-branes:

Assume Neumann boundary conditions in the directions $X^\mu\,,\mu = 0,1,\ldots,p$ and Dirichlet conditions in the directions $X^\nu\,,\nu = p+1,p+2,\ldots,d-1$.

Then the tensor product

$$|B(p-\text{brane})\rangle = |B(\tilde x_a^0)\rangle_N\otimes\ldots\otimes|B(\tilde x_a^p)\rangle_N\otimes|B(x_a^{p+1})\rangle_D\otimes\ldots\otimes|B(x_a^{d-1})\rangle_D$$

describes a D (Dirichlet) p-brane with fixed "dual locations" $\tilde x_a^\mu$ in the first $p+1$ directions and fixed locations x_a^ν in the last $d-1-p$ directions.

This is generalised to superstrings adding along with X free fermionic fields.

16.2. Open sector

$\hat u(1)\times\hat u(1)$ currents

$$\begin{aligned}
&J(z) = \sqrt{2}i\partial_z X(z,\bar z) = \sum a_n z^{-n-1}\,,\ \bar J(\bar z) = \sqrt{2}i\partial_{\bar z}X(z,\bar z) = \sum \bar a_n\bar z^{-n-1}\,,\\
&T(z) = \tfrac{1}{2}:JJ:(z)\,,\ \bar T(\bar z) = \tfrac{1}{2}:\bar J\bar J:(z)\\
&X(z,\bar z) = q - \frac{i}{2\sqrt{2}}(a_0+\bar a_0)\ln(z\bar z) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}(\frac{a_n}{n}z^{-n}+\frac{\bar a_n}{n}\bar z^{-n})\,,\\
&V_\alpha(z,\bar z) =: e^{2i\alpha X(z,\bar z)}:
\end{aligned} \tag{16.9}$$

Let (euclidean and Minkowski): $E_2 : \ z = e^{t+i\sigma} \, , \ \bar{z} = e^{t-i\sigma} \ \ M_2 : z = e^{i(\sigma^0+\sigma)} \, , \ \bar{z} = e^{i(\sigma^0-\sigma)}$
Bulk (half plane) free field - diagonal of $\hat{u}(1) \times \hat{u}(1)$

Neumann open b.c:

$$\partial_\sigma X(t,\sigma)_{|\sigma=0,\pi} = 0 \, , \leftrightarrow J(z) = \bar{J}(\bar{z})|_{z=\bar{z}} \tag{16.10}$$

solved by

$$X^H(z,\bar{z}) = q - \frac{i}{4}(a_0 + \bar{a}_0)\ln(z\bar{z}) + \frac{i}{2}\sum_{n\neq 0}\frac{a_n}{n}(z^{-n} + \bar{z}^{-n}) \tag{16.11}$$

the half-plane (bulk) current is $J(z,\bar{z}) = J(z) - \bar{J}(\bar{z}) = \sum_n a_n(z^n - \bar{z}^n)$.
For this condition

$$\langle V_\alpha(z,\bar{z})\rangle = \delta_{\alpha,0}$$

since $q^{i\alpha}$ cannot be compensated.

Dirichlet open b.c:

$$X(t,0) = x_a = \text{const} \ = X(t,\pi) \, , \text{or,} \ \ J(z) = -\bar{J}(\bar{z})|_{z=\bar{z}} \tag{16.12}$$

hence

$$X(z,\bar{z}) = x_a + \frac{i}{2}\sum_{n\neq 0}\frac{a_n}{n}(z^{-n} - \bar{z}^{-n})$$

(more generally, different boundaries at $\sigma = 0$ and $\sigma = \pi$, the constant term becomes $x_a + (x_b - x_a)\frac{\sigma}{\pi}$).

In this case there is a non-trivial 1-point function

$$\langle V_\alpha(z,\bar{z})\rangle_{x_a} = \frac{e^{2i\alpha x_a}}{(z-\bar{z})^{2h_\alpha}} \, , \quad h_\alpha = \alpha^2 \tag{16.13}$$

This is a $c = 1$ theory and (16.13) coincides up to the boundary dependent constant with the chiral 2-point function $\langle V_\alpha(z)V_{-\alpha}(\bar{z})\rangle$, where $V_\alpha(z) = e^{2\alpha i\phi(z)}$ with $\phi(z)$ as in (5.9).

- Dp - branes (in M_2) : $m = 0, 1, 2..., p$, (Neumann, world volume, parallel directions) while, $a = p+1, ...d-1$ (transverse , Dirichlet directions)

$$\begin{aligned} X^m &= \hat{x}^m + (2\alpha')\hat{p}^m\sigma^0 + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{a_n^m}{n}e^{in\sigma^0}\cos n\sigma^1 \, , \\ X^a &= x^a_{(i)} + (x^a_{(j)} - x^a_{(i)})\frac{\sigma^1}{\pi} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{a_n^m}{n}e^{in\sigma^0}\sin n\sigma^1 \, , \end{aligned} \tag{16.14}$$

If $i = 1, 2, ...N$ - this is interpreted as N Dp-branes located at positions $x^a_{(i)}$ of the target space.

Literature: e.g., [24].

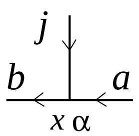
16.3. Boundary fields

The coordinate dependence of the 2 and 3-point correlators is determined by the invariance with respect to the real projective group $SL(2,\mathbb{R})$, so they differ by constants (and phases) from those of the chiral correlators.

$$\langle 0|{}^a\Psi^b_{i,\alpha_1}(x_1)\,{}^b\Psi^c_{j,\alpha_2}(x_2)\,{}^c\Psi^d_{k,\alpha_3}(x_3)|0\rangle_t = \delta_{ad}\frac{C^{abc}_{ijk\,;\alpha_1\,\alpha_2\,\alpha_3\,;t}}{|x_{12}|^{\triangle^k_{ij}}\,|x_{23}|^{\triangle^i_{jk}}\,|x_{31}|^{\triangle^j_{ki}}}\,, \tag{16.15}$$

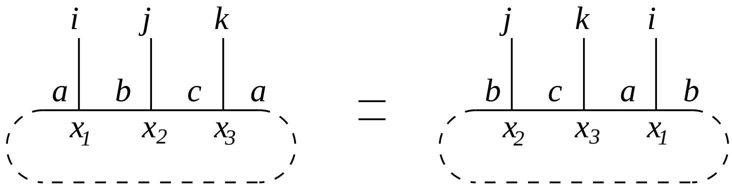
$$x_1 \neq x_2 \neq x_3 \neq x_1$$

In general $SL(2,\mathbb{R})$ representations are denoted by a pair $(\delta, \varepsilon = \pm)$, we choose $\varepsilon = 1$, corresponding to taking the modulus of the multiplier of the $SL(2,\mathbb{R})$ transformations and the above expression in (16.15) implies trivial monodromy of the boundary field correlators. In the WZW models the fields carry an additional tensor index, or, in a functional realisation, depend on an additional (multi)variable X. The boundary fields behave like chiral vertex operators but "intertwine" boundaries

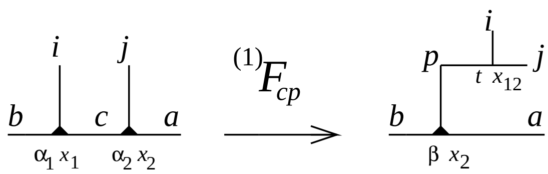
${}^b\Psi^a_{j,\alpha}(x) \;=$ 

with multiplicity α given by the NIM-reps solution $\alpha = 1, 2, \ldots, n^b_{ja}$.

The correlators are cyclically invariant - as on a disk or compactified line



We can expect that the n-point functions are some linear combinations of the chiral blocks - it will be made more precise; for the time being can be understood in terms of some formal projectors to boundaries.



The OPE of boundary fields is defined by a transformation matrix which now depends on three boundary and one standard CVO with multiplicity labels running over $\alpha_1 = 1, \cdots, n_{ic}{}^b$, $\alpha_2 = 1, \cdots, n_{ja}{}^c$, $\beta = 1, \cdots, n_{pa}{}^b$, $t = 1, \cdots, \mathcal{N}_{ij}{}^p$.

$$
\begin{aligned}
&{}^b\Psi^c_{i,\alpha_1}(x_1)\, {}^c\Psi^a_{j,\alpha_2}(x_2) \\
&\qquad = \sum_{p,\beta,t} {}^{(1)}F_{cp} \begin{bmatrix} i & j \\ b & a \end{bmatrix}^{\beta\ t}_{\alpha_1\,\alpha_2} \langle p,0|\phi^p_{ij;t}(x_{12})|j,0\rangle\ {}^b\Psi^a_{p,\beta}(x_2) + \ \ldots\ .
\end{aligned} \tag{16.16}
$$

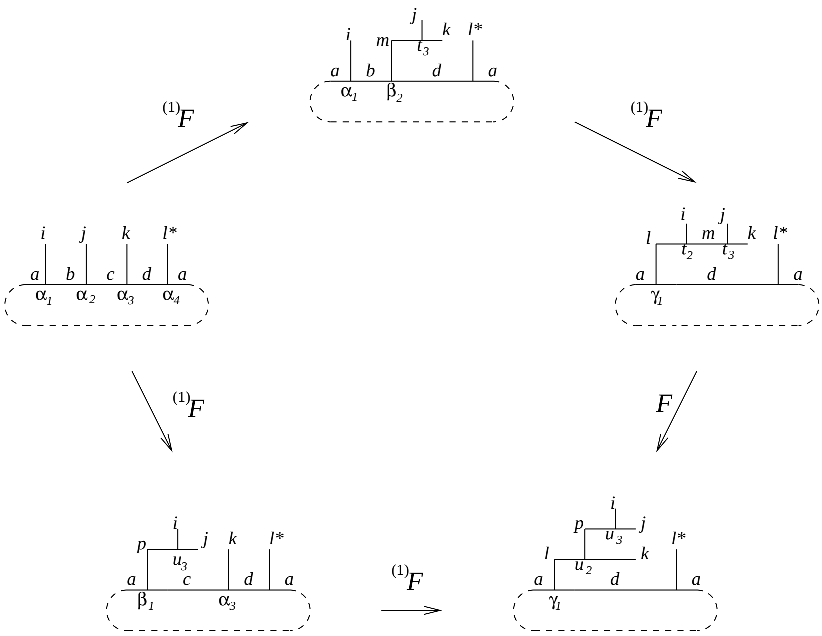
This transformation allows to convert any boundary fields correlator into a linear combination of ordinary chiral blocks - up to a normalisation of the 1-point function $< \mathbf{1} >_a$. In particular the 3-point function is proportional to the the matrix ${}^{(1)}F$, i.e., ${}^{(1)}F$ is essentially computing the 3-point boundary field constant, which now depends on 3 more parameters, the boundary parameters.

Denoting by $\mathcal{U}^b_{pa}$ the space of boundary fields of type $\binom{b}{p\,b}_{\alpha_1}$ we have dim $\mathcal{U}^b_{pa} = \nu_{pa}{}^b$ while the space of standard CVO, $\mathcal{V}^p_{ij}$, has dimension given by the Verlinde fusion multiplicity $\mathcal{N}_{ij}{}^p$, dim $\mathcal{V}^p_{ij} = \mathcal{N}_{ij}{}^p$. Thus we can interpret ${}^{(1)}F$ as a linear operator

$$
\oplus_c\, \mathcal{U}^b_{ic} \otimes \mathcal{U}^c_{ja} \to \oplus_p\, \mathcal{V}^p_{ij} \otimes \mathcal{U}^b_{pa}\,, \tag{16.17}
$$

the dimension of the two sides being identical, according to the NIM-rep eqn (15.20). The eqn (15.20) is again a necessary condition for the matrix ${}^{(1)}F$ to be invertible since the left and right sides count the number of states in both sides.

This second fusing matrix ${}^{(1)}F$ plays for the boundary fields one of the roles which the CVO fusing matric F plays - namely that of 3j symbol, defining the product. The associativity of the product (16.16) leads to a relation analogous to the pentagon equation but depending on two F - matrices



The "mixed" pentagon equation reads symbolically

$$F\ {}^{(1)}F\ {}^{(1)}F = {}^{(1)}F\ {}^{(1)}F \tag{16.18}$$

and in the diagonal case turns into the standard pentagon equation

$$F\ F\ F = F\ F \tag{16.19}$$

for the fusing matrices. This implies that we can identify ${}^{(1)}F$ with F in the diagonal case - eventually up to a gauge transformation - different for the boundary and the chiral vertices.

Literature: [66], [67], [65], [68].

17. Boundary CFT: half-plane bulk fields, sewing eqs, generalised CVO

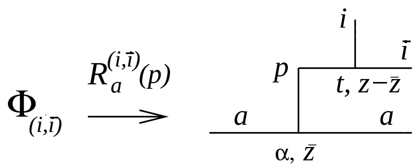
17.1. The half plane bulk fields

We have already determined the 1-point function of the half-plane bulk field. The invariance with respect to the subalgebra spanned by $L^{(H)}_{\pm 1,0}$ determines the 2–point function of a half-plane bulk and a boundary field

$$\begin{aligned}&\langle 0|{}^a\Psi^a_{p,\alpha}(x_1)\,\Phi^H_{(i,\bar{i})}\,(z,\bar{z})|0\rangle_t \\ &= \frac{C^a_{p,(i,\bar{i}),\alpha\,,t}}{(z-\bar{z})^{\triangle_i+\triangle_{\bar{i}}-\triangle_p}\,(x_1-z)^{\triangle_i+\triangle_p-\triangle_{\bar{i}}}\,(x_1-\bar{z})^{\triangle_{\bar{i}}+\triangle_p-\triangle_i}}\,,\quad x_1>\operatorname{Re} z\,,\end{aligned} \tag{17.1}$$

while $\langle\Phi_{(i,\bar{i})}\,{}^a\Psi^a_{p,\alpha}\rangle$ is defined for $\operatorname{Re} z > x_1$ by the analogous expression with x_1-z, $x_1-\bar{z}$ replaced by $z-x_1$, $\bar{z}-x_1$. As in the full plane we restrict to integer spin $\triangle_i-\triangle_{\bar{i}}\in\mathbb{Z}$ so that the expression does not depend on the order of the two fields - locality.

The 2-point function (17.1) is identical to a 3-point function of chiral vertex operators and in particular determines the OPE expansion of the two chiral fields, making the bulk field; this means $z\sim\bar{z}$, i.e., approaching the boundary. Effectively close to the boundary the bulk fields create boundary fields. The decomposition constant $R^I_a(p)$ - *bulk-boundary reflection coefficients* is related to the constant in (17.1)



$$= \sum_p \frac{R^{(i,\bar{i})}_a(p)}{(z-\bar{z})^{h_i+h_{\bar{i}}-h_p}}\,{}^a\Psi^a_{p,\alpha}(x)+\ldots \tag{17.2}$$

Which fields may appear is governed by the product of multiplicities $\mathcal{N}^p_{i\bar{i}}\,n^a_{pa}$.
For $p=1$ the reflection coefficient $R^I_a(1)$ gives the coefficient in the 1-point function (15.4), $R^{i,i^*}_a(1)<\mathbf{1}>=C_{h_i}$. It reduces to the boundary coefficient in (15.12),

$$R^{(i,i^*)}_a(1)e^{-i\pi h_i}=\frac{\psi^i_a}{\psi^1_a\,\sqrt{d_i}}\,. \tag{17.3}$$

This relation is obtained from the limit $L\to\infty$ of the 1-point correlator computed, as the partition function, in two alternative ways. For the partition function $Z_{a|a}$ itself the limit gives in the closed sector $\frac{\psi^1_a}{\sqrt{S_{11}}}\langle 0|\otimes\langle 0|a\rangle$ up to a power $\tilde{q}^{c/24}$, since the leading contribution is provided by the vacuum state. This allows to identify the 1-point boundary function

$$<\mathbf{1}>_a=\frac{\psi^1_a}{\sqrt{S_{11}}} \tag{17.4}$$

and similarly one obtains (17.3), normalising $C^1_{j^*j}=1$.

17.2. Bulk-boundary eqs

Besides the eqn (16.18) for the OPE coefficients ${}^{(1)}F$ of the boundary fields one can write a pair of bulk-boundary equations relating ${}^{(1)}F$ and the reflection coefficients in (17.2), first derived in the diagonal case by Lewellen [66]. The first of these equations is derived starting from the 3-point function of two boundary and one bulk operator, written in two ways

$$\langle {}^b\Psi^a_{j,\delta}(x_1)\,\Phi_{(i,\bar{i})}(z,\bar{z})\,{}^a\Psi^b_{k,\gamma}(x_2)\rangle = \langle {}^b\Psi^a_{j,\delta}(x_1)\,{}^a\Psi^b_{k,\gamma}(x_2)\Phi_{(i,\bar{i})}(z,\bar{z})\,\rangle \tag{17.5}$$

The mutual locality of the bulk and boundary fields implies the symmetry under the exchange of the bulk field with one of the boundary fields. Using in both expressions the decomposition (17.2) of the bulk field near the boundary and the boundary fields OPE of boundary fields. This results in different chiral blocks which are further related by braiding transformations. One obtains (omitting the fusion multiplicity indices),

$$\begin{aligned}
&\sum_{\beta} R^{(i,\bar{i})}_{b;\beta}(p)\ \sum_{\beta'} {}^{(1)}F_{b1}\begin{bmatrix} p^* & p \\ b & b \end{bmatrix}^{\mathbf{1}}_{\beta'\,\beta}\ {}^{(1)}F_{ap^*}\begin{bmatrix} j & k \\ b & b \end{bmatrix}^{\beta'}_{\delta\,\gamma} = \\
&\sum_{s\,,\,\alpha} R^{(i,\bar{i})}_{a,\alpha}(s)\sum_{\alpha'} {}^{(1)}F_{a1}\begin{bmatrix} s^* & s \\ a & a \end{bmatrix}^{\mathbf{1}}_{\alpha'\,\alpha}\ {}^{(1)}F_{bs^*}\begin{bmatrix} k & j \\ a & a \end{bmatrix}^{\alpha'}_{\gamma\,\delta} \times \\
&\sum_{m} e^{i\pi(2\triangle_i-2\triangle_m+\triangle_k+\triangle_j-\triangle_p)}\,F_{sm}\begin{bmatrix} j & i \\ k^* & \bar{i} \end{bmatrix} F_{mp}\begin{bmatrix} \bar{i} & i \\ k^* & j \end{bmatrix}
\end{aligned} \tag{17.6}$$

In the diagonal case we can choose $a=1$ which trivialises both ${}^{(1)}F$ functions and fixes $s=1\,,j=b=k^*\,,\bar{i}=i^*$ in the r.h.s. Then the r.h.s. reduces up to a constant to the Moore-Seiberg expression for the modular matrix $S(p)$ for the 1-point function on the torus, i.e., we can identify the bulk boundary constant with that modular matrix,

$$R^{(i,i^*)}_j(p) \quad = R^{(i,i^*)}_1(1)\,\frac{\sqrt{d_p}}{d_i\,d_j}\frac{S_{ij}(p)}{S_{11}} \tag{17.7}$$

In the last step we have used a gauge choice s.t., $F_{1p}\begin{bmatrix} \bar{i} & i^* \\ i & i \end{bmatrix} = \sqrt{d_p}/d_i$. Inserting (17.7) in the bulk-boundary equation (17.6) one reproduces the last of the Moore-Seiberg identities, relating $S(p)$ and the fusing (braiding) matrix. In the nondiagonal cases one can solve (17.6) in terms of $S(p)$ and ${}^{(1)}F$.

• The second of the equations represents the 2-point function of two half-plane bulk fields $\langle\Phi_{(k,\bar{k})}(z_1,\bar{z}_1)\Phi_{(l,\bar{l})}(z_2,\bar{z}_2)\rangle_a$ in two alternative ways - using either the bulk OPE or the decomposition into boundary fields (17.2). Once again the two decompositions reduce to chiral blocks related by braiding. In the diagonal case the resulting equation turns out to be equivalent to a relation, which is a consequence of the Moore-Seiberg identity relating $S(p)$ and B. The latter is reproduced by the equation (17.6) and thus in the diagonal case the two Lewellen bulk-boundary equations are not independent.

In the non-diagonal case the second bulk-boundary equation leads to an equation determining the bulk relative OPE coefficients. In general it involves both ${}^{(1)}F$ and F but it is simplified in the case when all three fields are scalars; then it reads

$$\frac{\psi_a^{(k,\alpha)}}{\psi_a^{\mathbf{1}}}\frac{\psi_a^{(l,\beta)}}{\psi_a^{\mathbf{1}}} = \sum_{j\in\mathcal{I},\gamma}\sum_t d_{(k,k;\alpha)(l,l;\beta)}^{(j,j;\gamma);t}\frac{\psi_a^{(j,\gamma)}}{\psi_a^{\mathbf{1}}} =: \sum_{(j,\gamma)\in\mathcal{E}} M_{(k,\alpha)\,(l,\beta)}{}^{(j,\gamma)}\frac{\psi_a^{(j,\gamma)}}{\psi_a^{\mathbf{1}}}\,. \tag{17.8}$$

Here $\psi_a^{(k,\alpha)}, k\in\mathcal{I}, \alpha = 1,2,\ldots,Z_{kk}$ is the unitary matrix (15.12) relating the set of boundary states $|a\rangle$ and the set of Ishibashi states. In the $sl(2)$ cases the ratios in the l.h.s. provide representations of the Pasquier algebra and hence the structure constants M in the r.h.s. coincide with the structure constants of that algebra: we recover the formula (11.9) for the OPE coefficients.

$$d_{(k,k;\alpha)(l,l;\beta)}{}^{(j,j;\gamma)} = \sum_a \frac{\psi_a^{(k,\alpha)}\psi_a^{(l,\beta)}\psi_a^{*,(j,\gamma)}}{\psi_a^{\mathbf{1}}} \tag{17.9}$$

Thus the BCFT interpretation provides a field theory derivation of the empiric result for the expression of the ADE relative scalar coefficients and allows to generalise it for higher rank in terms of a generalised Pasquier algebra, by definition spanned by the ratios of boundary state coefficients in the l.h.s. of (17.9).

17.3. Generalised CVO

The boundary fields can be represented in terms of the standard CVO and some interwtining operators of auxiliary finite dimensional spaces. Indeed, consider auxiliary vector spaces $V^j\cong\mathbb{C}^{m_j}$ with basis states $|e_{ba}^{j,\gamma}\rangle$, $\gamma = 1,2,\ldots,n_{ja}{}^b$, of dimension $m_j = \sum_{a,b} n_{ja}{}^b = \sum_{a,b,\gamma} 1$. An inner product in $\oplus_{j\in\mathcal{I}} V^j$ is defined as

$$\langle e_{ba}^{j,\gamma}|e_{b'a'}^{j',\gamma'}\rangle = \delta_{bb'}\delta_{aa'}\,\delta_{jj'}\,\delta_{\gamma'\gamma} \tag{17.10}$$

• We define the tensor product decomposition of states $|e^{i,\alpha}_{cb}\rangle \otimes |e^{j,\gamma}_{b'a}\rangle$ for coinciding $b' = b$ according to

$$|e^{i,\alpha}_{cb}\rangle \otimes_h |e^{j,\gamma}_{ba}\rangle = \sum_{k\in\mathcal{I}} \sum_{\beta=1}^{n_{ka}{}^c} \sum_{t=1}^{N_{ij}{}^k} {}^{(1)}F_{bk} \begin{bmatrix} i & j \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\beta\,t} |e^{k\,,\beta}_{ca}(ij;t)\rangle \,. \tag{17.11}$$

This is a "truncated" tensor product, in the sense that we restrict to a subspace $V^i \otimes_h V^j$ of $V^i \otimes V^j$, $(cb) \otimes_h (b'a) = \delta_{bb'}\ (cb) \otimes (b'a)$, with $\dim(V^i \otimes_h V^j) = \sum_{a,c} (n_i n_j)_a{}^c \leq m_i m_j$. The multiplicity of V^k in $V^i \otimes_h V^j$ is identified with the Verlinde multiplicity $N_{ij}{}^k$. Then the counting of states in both sides of (17.11) is consistent, taking into account (15.20). In (17.11) $e^{k\,,\beta}_{ca}(ij;t)$ give a basis, normalised as in (17.10), for the space V^k in

$$V^i \otimes_h V^j \cong \oplus_k N_{ij}{}^k\, V^k \ . \tag{17.12}$$

The ${}^{(1)}F \in \mathbb{C}$ are Clebsch-Gordan coefficients ("$3j$- symbols"). The completeness and orthogonality of the bases in $V^i \otimes_h V^j$ implies that ${}^{(1)}F_{bk}$ is an unitary matrix.

The requirement of associativity of the product (17.11) leads to the "mixed" pentagon relation (16.18) where F is the unitary matrix (the "$6j$-symbols"), relating the two bases in $V^i \otimes_h V^j \otimes_h V^k$ which satisfies the pentagon equation (16.19).

• We introduce the intertwining operator $V^j \to V^k$

$$P^{k,\alpha;j,\gamma}_{cb,ab} = |e^{k,\alpha}_{cb}\rangle\langle\, e^{j,\gamma}_{ab}| \,, \tag{17.13}$$

which corresponds to a state in $V^k \otimes_h V^{j^*}$. Now we can tensor the standard CVO with these projectors to define a a generalised chiral vertex operator (the dependence and the summation over the descendants is omitted)

$$\oplus_{j\in\mathcal{I}}\, \mathcal{V}_j \otimes V^j \to \oplus_{k\in\mathcal{I}}\, \mathcal{V}_k \otimes V^k \,,$$

$${}^c\Psi^a_{i,\beta}(z) = \sum_{j\,,k\,,t} \phi^k_{ij,t}(z) \otimes \sum_{b\,,\alpha\,,\gamma} {}^{(1)}F_{ak} \begin{bmatrix} i & j \\ c & b \end{bmatrix}_{\beta\,\gamma}^{\alpha\,t} P^{k,\alpha;j,\gamma}_{cb,ab} \,. \tag{17.14}$$

we have in particular

$${}^c\Psi^a_{j,\beta}(0)\,|0\rangle \otimes |e^1_{aa}\rangle = \phi^j_{j1}(0)|0\rangle \otimes |e^{j,\beta}_{ca}\rangle =: |j\,,\beta\rangle \,, \quad \beta = 1, 2 \ldots, n_{ja}{}^c \tag{17.15}$$

where $|j\,,\beta\rangle$ is the explicit form of the highest weight state of the chiral algebra representation $\mathcal{V}_{j,\,\beta}$, used in the computation of the cylinder partition function (15.6) in the Hilbert space $\mathcal{H}_{ac}$. For real arguments one recovers the correlators of the boundary fields. More precisely, to reproduce also the standard normalisation (17.4), one has to use different normalisations in (17.10) and (17.11), which we have skipped here in order to simplify the presentation.

• For the generalised CVO (17.14) one can define a new braiding transformation $\hat{B}$, diagonalised by $\overset{(1)}{F}$

$$\hat{B}_{bd} \begin{bmatrix} i & j \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\alpha'\,\gamma'} (\epsilon) = \sum_{k\,,\beta\,,t} \overset{(1)}{F}_{bk} \begin{bmatrix} i & j \\ c & a \end{bmatrix}_{\alpha\,\gamma}^{\beta\;t} \; e^{-i\pi\epsilon\,\triangle_{ij}^{k}} \; \overset{(1)}{F}{}^{*}_{dk} \begin{bmatrix} j & i \\ c & a \end{bmatrix}_{\alpha'\,\gamma'}^{\beta\;t} . \tag{17.16}$$

This formula determines $\hat{B}$ whenever we know $\overset{(1)}{F}$ and the scaling dimensions $\triangle_j$. The matrix $\hat{B}$ satisfies a set of new polynomial relations involving $F, \overset{(1)}{F}, B$, like e.g. the relation written symbolically as

$$\overset{(1)}{F}\,\overset{(1)}{F}\,B = \hat{B}\,\overset{(1)}{F}\,\overset{(1)}{F}\,,$$

in which $\overset{(1)}{F}$ may be interpreted as intertwining the two representations B and $\hat{B}$ of the braiding group.

• Let now $I = (i, \bar{i}; \alpha)$, $i\,, \bar{i} \in \mathcal{I}$, $\alpha = 1, 2, \ldots Z_{j\bar{j}}$ label the "physical" spectrum, corresponding to nonzero matrix elements of the modular matrix $Z_{i\bar{i}}$. The (upper) half-plane bulk fields $\Phi^{H}_{(i,\bar{i}^*;\alpha)}(z, \bar{z})$ are defined as compositions of two GCVO (17.14)

$$\begin{aligned} \Phi^{H}_{(i,\bar{i}^*;\alpha)}(z, \bar{z}) &= \sum_{a,b\,,\beta'\,,\beta} \Big(\sum_{j,\alpha,u} R^{(i,\bar{i}^*,u)}_{a,\alpha}(j)\, \overset{(1)}{F}{}^{*}_{bj} \begin{bmatrix} i & \bar{i}^* \\ a & a \end{bmatrix}_{\beta\,\beta'}^{\alpha\,u} \Big)\, {}^{a}\Psi^{b}_{i,\beta}(z)\ {}^{b}\Psi^{a}_{\bar{i}^*,\beta'}(\bar{z}) \\ &= \sum_{n\,,k\,,l\,,t\,,t'} \phi^{n}_{ik;t'}(z)\, \phi^{k}_{\bar{i}^*l;t}(\bar{z}) \otimes \sum_{a\,,b'\,,\gamma\,,\gamma'} C^{n,k,l;t',t}_{(i,\bar{i})\,a,b',a;\gamma,\gamma'}\ P^{n,\gamma;l,\gamma'}_{ab',ab'}\,. \end{aligned} \tag{17.17}$$

Here $\bar{z} \in H_-$ is the complex conjugate of $z \in H_+$. The coefficients C are expressed in terms of R, F and $\overset{(1)}{F}$ using (17.14). For small $z - \bar{z} = 2i\,y$ using the OPE (16.16) for the two CVO in (17.17) (and projecting on $|e^{1}_{aa}\rangle\rangle$) we recover in the leading order the boundary field ${}^{a}\Psi^{a}_{j,\alpha}(x)$ contributing with the bulk-boundary reflection coefficient $R^{(i,\bar{i}^*,u)}_{a,\alpha}(j)$.

• All the equations of the boundary CFT are then rederived; in fact some more relations arise from the unitarity of $\overset{(1)}{F}$, which is not an intrinsic property in the framework of the boundary CFT.

The construction above is inspired by the quantum algebra proposed by A. Ocneanu as a generalisation of quantum groups, to be discussed in the next lecture.

Literature: [66], [69], [65], [70].

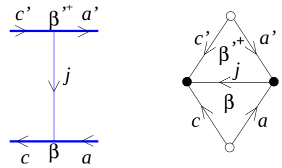
18. Conformal interfaces (defects), Ocneanu quantum algebras

- Motivation - two-fold:

  a) Find a closed expression for the OPE coefficients for local fields of non-trivial spin.

  b) Understand relation to work of Ocneanu:

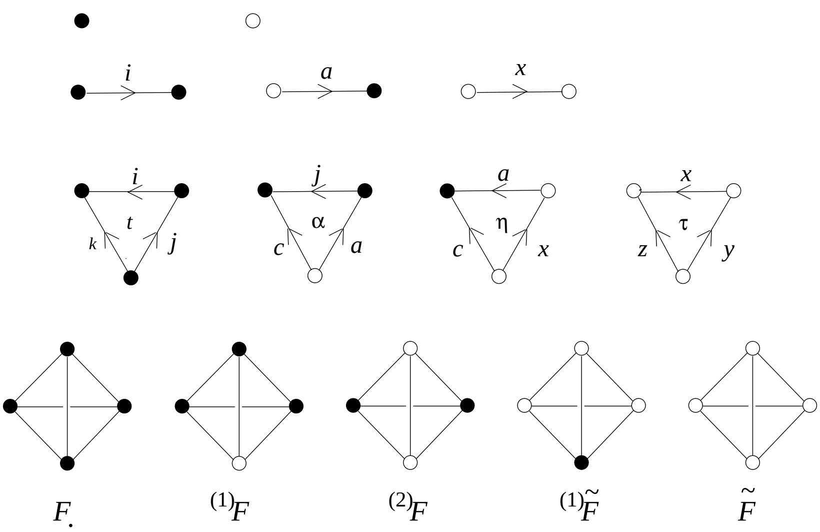
- BCFT data fits into the combinatorial data defining one of the pair of dual (block) matrix algebras $\mathcal{A}, \tilde{\mathcal{A}}$, a structure introduced by A. Ocneanu [71], [72]. It is known as "double triangle algebra" (DTA), or, loosely, quantum groupoid, see also [73], [74] for a realisation in the subfactor theory. The same structure has also a more axiomatic formulation [75] - as "weak C^*-Hopf algebras" (WHA). The DTA $\mathcal{A}$ is an algebra with two multiplications- the second corresponds to the "horisontal" tensor product we have discussed, with $\mathcal{A} = \oplus_{j\in\mathcal{I}} M_{m_j}$ realised as the space $\oplus_{j\in\mathcal{I}}$ End(V^j), where each block is represented by matrices corresponding to states in $V^j \otimes V^{j^*}$, $e^{(ca),(c'a')}_{j;\beta,\beta'} \sim |e^{j,\beta}_{ca}\rangle\langle e^{j,\beta'}_{c'a'}|$, depicted as "double triangles",



In the WHA approach one defines instead a coproduct.

- The two "fusing" matrices ${}^{(1)}F$ and F can be identified with the 3j- and the 6j- symbols of the WHA, while the pair of pentagon equations (16.18) and (16.19), are part of the "Big Pentagon" equation of the DTA/WHA. However these algebras involve much more structures, no trace of which is seen in the CFT, or BCFT.

The combinatorial data is summarised in a 3-simplex



- Comparing with the boundary CFT we recognise 2-types of vertices, two multiplicities, $\mathcal{N}_{ij}^{k}$ and n_{ja}^{b}, two pairs of tetrahedra ${}^{(1)}F$ and F - the 3j and 6 j -symbols

  What about the dual quantities? What is their physical interpretation?

- Recall that the $sl(2)$ Dynkin diagrams G and their generalisations are determined from the diagonal $j = \bar{j}$ of the matrix $Z_{j\bar{j}}$ in the modular invariant (11.2), and encode in fact a solution of the NIM-rep equation, a representation of the Verlinde fusion algebra, related to the chiral half of the theory. Accounting for the off-diaginal spectrum determined by the off-diagonal elements of Z leads to new graphs, the Ocneanu graphs $\tilde{G}$, first constructed by him in the $\widehat{sl}(2)$ case

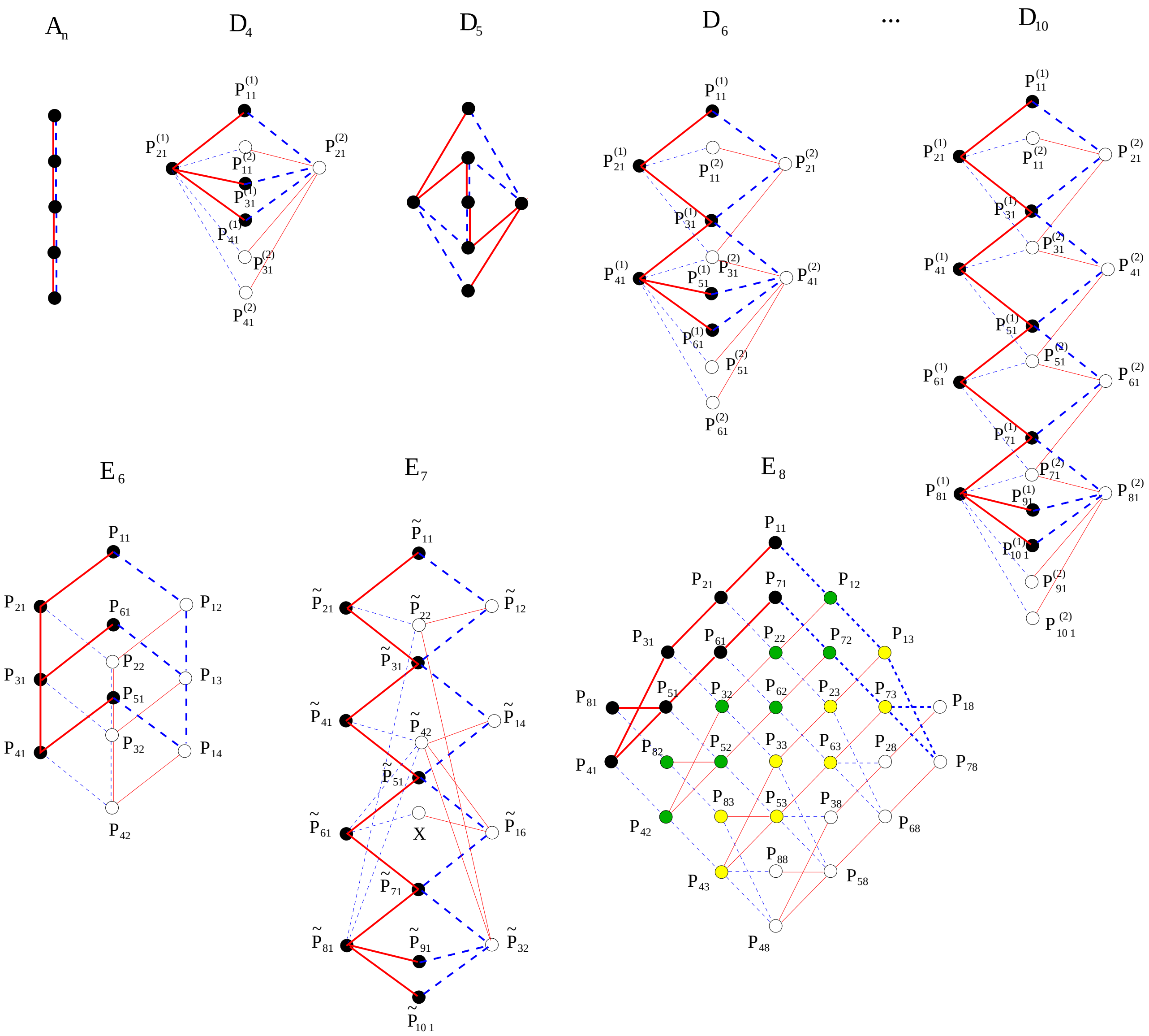


Fig. 1: $sl(2)$ Ocneanu graphs

- The main idea is to generalise Cardy eqn to the full non-diagonal spectrum - result : partial answer about the physical interpretation of the dual DTA - so far, on the level of the new multiplicities.

18.1. Torus partition functions with twisted boundary conditions

We assume that the modular invariant partition functions are known, i.e., the multiplicities $Z_{j,\bar{j}}$ are given. These partition functions were computed first considering the system on a cylinder, we may think of the torus as made of two cylinders of the boundary case. Imposing periodic boundary conditions in the imaginary w direction $w+iL = w-iL$ one obtains a torus and computes the trace of the operator $e^{-2LH^{cyl}}$. Now we shall allow the possibility of inserting one (or several) operator(s) X inside the trace of this evolution operator. This may be interpreted as introducing one or several defect lines $\mathcal{C}$ (" seams") in the system, along non- contractible cycles of the cylinder, before closing it into a torus, thus resulting into a certain class of non-periodic, "twisted" boundary conditions. In fact at the end of this more abstract construction one does recover some known examples of such non-periodic system, considered both in microscopic level and in relation to CFT.

The operators X are not arbitrary, they are assumed to commute with the energy-momentum tensor $T(w), \bar{T}(\overline{w})$, or equivalently with the Virasoro generators

$$[L_n, X] = [\bar{L}_n, X] = 0\ . \tag{18.1}$$

Since the Virasoro operators are the generators of infinitesimal diffeomorphisms, this condition says that each operator X is invariant under a distorsion of the line to which it is attached.

We shall first restrict ourselves to operators intertwining a pair of components of (11.1), i.e. mapping some $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$ into $\mathcal{V}_{j'} \otimes \overline{\mathcal{V}}_{\bar{j}'}$: irreducibility of the representations $\mathcal{V}_j$ tells us that such an X is non trivial only for $j = j'$, $\bar{j} = \bar{j}'$. If the multiplicity $Z_{j\bar{j}}$ is 1, it follows that X must be proportional to the projector $P^j \otimes P^{\bar{j}}$ in $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$. If however $Z_{j\bar{j}} > 1$, X is a linear combination of operators intertwining the different copies of $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$

$$P^{(j,\bar{j};\alpha,\alpha')}\ :\ (\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha')} \to (\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha)} \quad \alpha, \alpha' = 1, \cdots, Z_{j\bar{j}}\ , \tag{18.2}$$

and acting as $P^j \otimes P^{\bar{j}}$ in each. If $|j, \mathbf{n}\rangle \otimes |\bar{j}, \bar{\mathbf{n}}\rangle$ denotes an orthonormal basis of $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}}$ labelled by multi-indices $\mathbf{n}, \bar{\mathbf{n}}$, we may write

$$P^{(j,\bar{j};\alpha,\alpha')} = \sum_{\mathbf{n},\bar{\mathbf{n}}} \big(|j,\mathbf{n}\rangle \otimes |\bar{j},\bar{\mathbf{n}}\rangle\big)^{(\alpha)} \big(\langle j,\mathbf{n}| \otimes \langle \bar{j},\bar{\mathbf{n}}|\big)^{(\alpha')} \quad \alpha,\alpha' = 1,\cdots Z_{j\bar{j}} \ . \tag{18.3}$$

There are thus $\sum_{j,\bar{j}} |Z_{j\bar{j}}|^2$ linearly independent solutions of equations (18.1). If these equations are extended to the generators of the full chiral algebra $\mathfrak{A}$, there may be more general solutions.

The P's satisfy

$$P^{(j_1,\bar{j}_1;\alpha_1,\alpha'_1)} P^{(j_2,\bar{j}_2;\alpha_2,\alpha'_2)} = \delta_{j_1 j_2}\delta_{\bar{j}_1\bar{j}_2}\delta_{\alpha'_1\alpha_2}\, P^{(j_1,\bar{j}_1;\alpha_1,\alpha'_2)} \ . \tag{18.4}$$

They play here the rôle of the Ishibashi states in the problem of boundary conditions in the half plane. We then write the most general linear combination of these basic operators as

$$X_x = \sum_{j\bar{j},\alpha,\alpha'} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')}}{\sqrt{S_{1j}S_{1\bar{j}}}}\, P^{(j,\bar{j};\alpha,\alpha')} \ , \tag{18.5}$$

with x a label taking $n = \sum_{j,\bar{j}} (Z_{j\bar{j}})^2$ values and Ψ an a priori arbitrary complex $n \times n$ matrix. The denominator $\sqrt{S_{1j}S_{1\bar{j}}}$ is introduced for later convenience. We shall denote by $\widetilde{\mathcal{V}}$ the set of labels x and use the label $x = 1$ for the identity operator

$$X_1 := \mathrm{Id} = \sum_{j\bar{j},\alpha} P^{(j,\bar{j};\alpha,\alpha)} \ , \tag{18.6}$$

for which

$$\Psi_1^{(j,\bar{j};\alpha,\alpha')} = \sqrt{S_{1j}S_{1\bar{j}}}\,\delta_{\alpha\alpha'} =: \Psi_1^{(j,\bar{j})}\,\delta_{\alpha\alpha'} \ . \tag{18.7}$$

Using (18.4) and the hermitian conjugation properties of the projectors

$$(P^{(j,\bar{j};\alpha,\alpha')})^\dagger = P^{(j,\bar{j};\alpha',\alpha)} \tag{18.8}$$

two such X are composed as

$$X_y^\dagger\, X_x = \sum_{j,\bar{j},\alpha,\alpha',\beta} \frac{\Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}\ \Psi_x^{(j,\bar{j};\alpha,\beta)}}{S_{1j}S_{1\bar{j}}}\, P^{(j,\bar{j};\alpha',\beta)} \ . \tag{18.9}$$

where $\Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}$ is the complex conjugate of $\Psi_y^{(j,\bar{j};\alpha,\alpha')}$.

As in the previous discussion it is convenient to map the cylinder into the complex plane with coordinate ζ by $\zeta = \exp -2i\pi\frac{w}{T}$. The toroidal domain is mapped into an annulus with identified boundaries along the circles $|\zeta| = |\tilde{q}|^{\pm 1/2}$; here $\tilde{\tau} = 2iL/T$ and

$\tilde{q} = \exp 2i\pi\tilde{\tau}$. One then reexpresses the partition function in terms of Virasoro generators acting in that plane using once again that

$$Z_{y|x} = \mathrm{tr}_{\mathcal{H}_P}(X_y^+ \, X_x \, e^{-2LH^{cyl}}) = \mathrm{tr}_{\mathcal{H}_P}(X_y^+ \, X_x \, \tilde{q}^{L_0-c/24} \, \tilde{q}^{\bar{\Lambda}_0-c/24}) \ , \tag{18.10}$$

With the help of

$$\mathrm{tr}_{\mathcal{H}_P}(P^{(j,\bar{j};\alpha,\alpha')} \tilde{q}^{L_0-c/24} \, \tilde{q}^{\bar{L}_0-c/24}) = \chi_j(\tilde{q}) \, \chi_{\bar{j}}(\tilde{q}) \; \delta_{\alpha\alpha'} \ , \tag{18.11}$$

and of (18.8) we write the corresponding twisted partition function as

$$Z_{y|x} := Z_{X_y^\dagger \, X_x} = \sum_{\substack{j,\bar{j}\in\mathcal{I} \\ \alpha,\alpha'=1,\cdots,Z_{j\bar{j}}}} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')} \Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}}{S_{1j} S_{1\bar{j}}} \; \chi_j(\tilde{q}) \, \chi_{\bar{j}}(\tilde{q}) \, . \tag{18.12}$$

In particular, for $x = y = 1$, we find

$$Z_{1|1} = \sum_{j,\bar{j},\alpha} \chi_j(\tilde{q}) \, \chi_{\bar{j}}(\tilde{q}) = \sum_{j,\bar{j}\in\mathcal{I}} Z_{j\bar{j}} \, \chi_j(\tilde{q}) \, \chi_{\bar{j}}(\tilde{q}) \ , \tag{18.13}$$

which is the modular invariant partition function describing the system with no twist. This can be extended to unspecialised characters and complex τ so that so that in (18.12) and in (18.13) the character $\chi_{\bar{j}}(\tilde{q})$ is replaced by complex conjugated character $\chi_{\bar{j}}(\tilde{q})^*$.

Because of the identification of its two ends, the cylinder considered above may be mapped into another plane, with coordinate $z = \exp(\pi \frac{w}{L})$. The image of the fundamental domain in w is an annulus in that plane with boundaries along the circles $|z| = 1$ and $|z| = |q|^{-1}$ identified, with now $q = \exp 2i\pi\tau$, $\tau = -1/\tilde{\tau} = iT/2L$. On the cylinder, one may also use the Hamiltonian corresponding to the Re w-translation operator, a combination of $T(w), \bar{T}(\overline{w})$ which is expressed as the two plane Vir modes $L_0 + \bar{L}_0 - \frac{c}{12}$. Then the partition function $Z_{x|y}$ is obtained as the trace of the corresponding evolution operator in a Hilbert space in the plane which depends on the presence of the defects,

$$\mathcal{H}_{y|x} = \oplus_{i,\bar{i}\in\mathcal{I}} \, \widetilde{V}_{i\bar{i}^*;\,x}{}^y \, \mathcal{V}_i \otimes \overline{\mathcal{V}}_{\bar{i}} \ , \tag{18.14}$$

Here $\widetilde{V}_{i\bar{i};\,x}{}^y$ is a non-negative integer, the multiplicity with which a possible pair of representations of the two algebras appears in the presence of the defects x and y. In the trivial case $x = y = 1$, it must reduce to

$$\widetilde{V}_{i\bar{i}^*;\,1}{}^1 = Z_{i\bar{i}} \ . \tag{18.15}$$

We can thus complete the calculation as in the absence of the X operators and get (shifting to unspecialised characters)

$$Z_{x|y} = \mathrm{tr}_{\mathcal{H}_{x|y}}\, q^{L_0-c/24}\,\bar q^{\bar L_0-c/24} = \sum_{i,\bar i\in\mathcal{I}} \widetilde{V}_{i\bar i^*;\,x}{}^{y}\ \chi_i(q)\,\chi_{\bar i}(q)^*\ . \tag{18.16}$$

Identifying the two expressions (18.12) and (18.16) and using the modular transformations we get

$$\widetilde{V}_{i\bar i;\,x}{}^{y} = \sum_{j,\bar j,\alpha,\alpha'} \frac{S_{ij}S_{\bar i\bar j}}{S_{1j}S_{1\bar j}}\ \Psi_x^{(j,\bar j;\alpha,\alpha')}\ \Psi_y^{(j,\bar j;\alpha,\alpha')\,*}\,, \qquad i,\bar i\in\mathcal{I}\,, \tag{18.17}$$

an analog of Cardy eqn, a new consistency condition on the matrices Ψ and the set of non-negative integers $\widetilde{V}_{i\bar i;\,x}{}^{y}$, restricted by the initial condition (18.15). The numbers $\widetilde{V}_{i\bar i;x}{}^{y}$ can be regarded either as the entries of $|\mathcal{I}|\times|\mathcal{I}|$ matrices $\widetilde{V}_x{}^y$, $x,y\in\widetilde{\mathcal{V}}$, or as those of $|\widetilde{\mathcal{V}}|\times|\widetilde{\mathcal{V}}|$ matrices $\widetilde{V}_{i\bar i}$, $i,\bar i\in\mathcal{I}$.

• We now make the additional assumption that the $\Psi_x^{(j,\bar j;\,\alpha,\alpha')}$ form a unitary (i.e. orthonormal and complete) change of basis from the $P^{(j,\bar j,\alpha,\alpha')}$ to the X_x operators. Equation (18.17) then may be regarded as the spectral decomposition of the matrices $\widetilde{V}_{i\bar i}$ into their orthogonal eigenvectors Ψ and eigenvalues $\frac{S_{ij}}{S_{1j}}\,\frac{S_{\bar i\bar j}}{S_{1\bar j}}$; in particular $V_{11}=\mathbf{1}$ is the identity matrix. Using that these ratios of the modular matrices S form a representation of the tensor product of two copies of Verlinde fusion algebra, the same holds true for the $\widetilde{V}$ matrices:

$$\widetilde{V}_{i_1j_1}\widetilde{V}_{i_2j_2} = \sum_{i_3,j_3} N_{i_1i_2}{}^{i_3}N_{j_1j_2}{}^{j_3}\ \widetilde{V}_{i_3j_3}\,,\quad i_k,j_k\in\mathcal{I}. \tag{18.18}$$

Thus in full analogy with the boundary problem, the problem of describing a complete set of "defects" is reduced to the solution of this new NIM-reps equation for the commuting matrices $\widetilde{V}$, with the constraint (18.15). (It plays a similar role of an initial input as do the branching coeffs $n_{i1}{}^{b}$ in the analogous boundary problem.) Combining (18.15) with (18.18), we have in particular

$$\sum_{i_3j_3} N_{i_1i_2}{}^{i_3}N_{j_1j_2}{}^{j_3}\ Z_{i_3j_3} = \sum_{x} \widetilde{V}_{i_1j_1^*;\,1}{}^{x}\widetilde{V}_{i_2j_2^*;\,x}{}^{1}\ , \tag{18.19}$$

which is the way the matrices $\widetilde{V}_{ij;\,1}{}^{x}=\widetilde{V}_{i^*j^*;\,x}{}^{1}$ appeared originally in the work of Ocneanu. All $\widetilde{V}_x{}^y$ may be reconstructed from the simpler Ocneanu matrices $\widetilde{V}_1{}^x$. A conjugation in the set $\widetilde{\mathcal{V}}$ is defined by

$$\Psi_{x^*}^{(J;\,\alpha,\beta)} = \Psi_x^{(J;\,\beta,\alpha)\,*} \tag{18.20}$$

• Note that for $j_1=j_2=1$, or $i_1=i_2=1$, we get a representation of one Verlinde algebra, but with a matrix of bigger size than $\mathcal{E}$; these matrices are reducible and they may contain n_i as submatrices.

18.2. Solutions and examples

For the diagonal theories, where $Z_{j\bar{j}} = \delta_{j\bar{j}}$, the cardinality of the set of defects is that of $\mathcal{I}$ and we can identify the two sets. Then the equation (18.18) is solved for

$$\widetilde{V}_{ij} = N_i N_j \qquad (18.21)$$

understood as a matrix product, in particular $\widetilde{V}_{ij;\,1}{}^k = N_{ij}{}^k$. The corresponding $\Psi_x^{(j,j)}$ are just the modular matrix elements S_{xj}.

Example:
Consider the diagonal case (A_2, A_3), of the minimal theories, i.e., Ising model, with Vir irreps labelled by the central charge $c = \frac{1}{2}$ and the dimensions $h_1 = 0\,, h_2 = \frac{1}{16}\,, h_3 = \frac{1}{2}$. The twisted partition functions $Z_{0|x}$ are three,

$$\begin{aligned}
&Z_{1|1} = |\chi_1(\tilde{q})|^2 + |\chi_3(\tilde{q})|^2 + |\chi_2(\tilde{q})|^2 = |\chi_1(q)|^2 + |\chi_3(q)|^2 + |\chi_2(q)|^2 \\
&Z_{1|3} = |\chi_1(\tilde{q})|^2 + |\chi_3(\tilde{q})|^2 - |\chi_2(\tilde{q})|^2 = \chi_1(q)\chi_3(q)^* + \text{c.c.} + |\chi_2(q)|^2 \\
&Z_{1|2} = \sqrt{2}(|\chi_1(\tilde{q})|^2 - |\chi_3(\tilde{q})|^2) = \big(\chi_1(q) + \chi_3(q)\big)\chi_2(q)^* + \text{c.c.} \\
&\Rightarrow\ Z_{1|1} = Z_{3|3}\,, \\
&Z_{2|2} = Z_{1|1} + Z_{1|3} = 2(|\chi_1(\tilde{q})|^2 + |\chi_3(\tilde{q})|^2) = |\chi_1(q) + \chi_3(q)|^2 + 2|\chi_2(q)|^2
\end{aligned} \qquad (18.22)$$

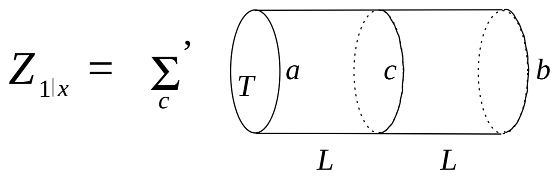
The first of these partition functions is the modular invariant one obtained as a system with periodic boundary conditions, or $V^1_{ij^*;1} = N^1_{ij^*} = \delta_{ij}$. The second $Z_{1|3} = Z_{1|\sigma(1)}$ reproduces the simplest of the torus partition functions with $\mathbb{Z}_2$ twisted (periodic and anti-periodic) boundary conditions [76], and is an example of a defect related to a group, here $V^{\sigma(1)}_{ij;1} = N^{\sigma(3)}_{ij} = \delta_{\sigma(i)j}$. This partition function describes the operator content of quasi-local correlators, with half-spin operators appearing in the OPE of an order and disorder scalar operator, both of dimension $\frac{1}{16}$; this can be now interpreted as an OPE of one scalar operator in the presence of a defect. The third partition function in (18.22) appears to correspond to the critical point of an Ising quantum chain with "duality- twisted" boundary conditions [77].

- As a second case, consider a non-diagonal theory with a matrix $Z_{ij} = \delta_{i\zeta(j)}$, where ζ is the conjugation of representations or some other automorphism of the fusion rules (like the Z_2 automorphism in the $D_{2\ell+1}$ cases of $\widehat{sl}(2)$ theories). Then $\widetilde{V}_{ij} = N_i N_{\zeta(j)}$ and $\Psi_i^j = S_{\zeta(i),j}$.

• In particular in the case $\widehat{sl}(2)$ there are compact solutions for the block-diagonal cases, $D_{\rm even}, E_6, E_8$, which generalise (18.21); it is illustrated on the picture below. The matrices $\widetilde{V}_{ij;\,1}{}^x$ can be expressed simply as bilinear combinations of the NIM-reps matrices $n_{ia}{}^b$ where the indices a, b belong to the set $\mathcal{V}$ of vertices of the Dynkin diagram, and parametrise the possible boundary conditions. The labels x may be taken of the form (a, b, γ), $a, b \in \mathcal{V}$, γ an extra label, and

$$\widetilde{V}_{ij^*;\,1}{}^{(a,b,\gamma)} = \sum_{c\in T_\gamma} n_{ic}{}^a n_{jc}{}^b \tag{18.23}$$

with c running over a certain subset T_γ of vertices; this representation generalises that for the modular matrix, recovered for $a = b = 1$



• For the $D_{2\ell}$ case, we take $b = 1$, $\gamma = 0, 1$, i.e., $\{x = (a, \gamma)\}$ and T_γ is the set of vertices of $\mathbb{Z}_2$ grading equal to γ.

• For the conformal embedding cases E_6, resp. E_8, $b = 1, 2$, resp $b = 1, 2, 3, 8$, the label γ is dropped, and the range of summation of c is the subset $T \subset \mathcal{V}$, identified with the set of representations of the extended fusion algebra, i.e., $T = \{1, 5, 6\}$, resp. $\{1, 7\}$. (For $D_{2\ell}$, $T = T_{\gamma=0}$.)

• Finally, the case E_7 is exceptional, as all but one of the matrices $\widetilde{V}_1{}^x$ may be represented by a formula similar to (18.23), in terms of the n matrices of the "parent theory" D_{10}.

• Eqn (18.23) provides closed expressions for the matrices $\widetilde{V}_{ij;1}{}^x$ already known from the work of Ocneanu. To any of these $\widehat{sl}(2)$ cases Ocneanu has associated a graph $\widetilde{G}$ with a set of vertices given by $\widetilde{\mathcal{V}}$; Ocneanu graphs were displayed in the beginning of this section. The graphs are generated by the pair of adjacency matrices, $\widetilde{V}_{2,1}$ and $\widetilde{V}_{1,2}$.

• For the minimal $c < 1$ theories of central charge $c = 1 - 6(g-h)^2/gh$ classified by the pair (A_{h-1}, G), with h odd, and g even, the Coxeter number of G, the set $\mathcal{I}$ of Virasoro representations (r, s) is restricted by $1 \le r \le h-1$, $1 \le s \le g-1$ and we choose r odd. Then the twist labels are of the form (r, x), x- a twist label of the corresponding non-diagonal $\widehat{sl}(2)_{g-2}$ theory labelled by G,

$$\widetilde{V}_{(rs)(r's');\,1}{}^{(r'',x)} = \mathcal{N}_{rr'}{}^{r''}\ \widetilde{V}^{(G)}_{ss';\,1}{}^{x}\ , \qquad r, r', r''\ \text{odd}\ , \tag{18.24}$$

in terms of the fusion matrices $\mathcal{N}_r$ of $\widehat{sl}(2)_{h-2}$ and of the $\widetilde{V}^{(G)}$ matrices of the G case.

Literature: [71], [72], [78], [70].

19. Defects on the cylinder, fusion algebra of defects and general OPE coefficients

19.1. The Ocneanu graph algebra $\widetilde{N}$

In the diagonal case formula (18.21) implies that $\widetilde{V}_y{}^z$ are linear combinations of $\widetilde{V}_1{}^x$, i.e., $\widetilde{V}_x{}^z = \sum_x N_{yx}{}^z \widetilde{V}_1{}^y$. This formula generalises to other cases, the Verlinde matrix being replaced by a new nonnegative integer valued matrix $\widetilde{N}_{xy}{}^z$. In terms of the partition functions we have

$$Z_{z|x} = \sum_y \widetilde{N}_{yx}{}^z \; Z_{y|1} \tag{19.1}$$

Equivalently, the operators (18.5) close an algebra - the *fusion algebra of defects*

$$X_y X_x = \sum_z \widetilde{N}_{yx}{}^z \, X_z \,, \tag{19.2}$$

which can be also expressed as a trace

$$\widetilde{N}_{yx}{}^z = \mathrm{Tr}(X_y \; X_x \; X_z^\dagger) \tag{19.3}$$

for $\mathrm{Tr} P^{(J;\,\alpha,\beta)} := \delta_{\alpha\beta}\, S_{1j}\, S_{1\bar{j}}$; this definition of the trace may be justified in unitary CFT's in exactly the same way as the norm of the Ishibashi states, via the $\tau \to \infty$ asymptotics of the characters $\chi_j(\tau)$.

More generally we require that the coefficients in the partition function for arbitrary number of defects are multiplicities, i.e., non-negative integers

$$\begin{aligned}
\widetilde{V}_{ik;y_1,\dots y_{n-1}}{}^{y_n} &= \sum_{j,\bar{j},\alpha,\alpha'} \frac{S_{ij} S_{k\bar{j}}}{S_{1j} S_{1\bar{j}}} \; \frac{\Psi_{y_1}^{(j,\bar{j};\alpha_1,\beta_1)} \; \dots \Psi_{y_n}^{(j,\bar{j};\alpha_1,\beta_n)\,*}}{(S_{1j} S_{1\bar{j}})^{n-2}} \\
\widetilde{V}_{ij;1,\dots y_{n-1}}{}^{y_n} &= \widetilde{V}_{ij;y_2,\dots y_{n-1}}{}^{y_n} \,, \\
\widetilde{V}_{11;y_1,\dots y_{n-1}}{}^{y_n} &= \widetilde{N}_{y_1,\dots y_{n-1}}{}^{y_n} = \mathrm{Tr}(X_{y_1} \dots X_{y_{n-1}} X^{+}_{y_n}) \,, \; \widetilde{V}_{11;y_1}{}^{y_2} = \delta_{y_1 y_2}
\end{aligned} \tag{19.4}$$

then we will have an analog of the `NIM`-rep but with different n

$$\widetilde{V}^{(n)}_{i_1 j_1} \widetilde{V}^{(m)}_{i_2 j_2} = \sum N_{i_1 i_2}^{i_3} N_{j_1 j_2}^{j_3} \widetilde{V}^{(n+m-2)}_{i_3 j_3}$$

Taking $i_2 = 1 = j_2 \,, y_1 = 1$ and $m = 3 \,, n = 2$ we recover the relation (19.1)

$$\sum_y \widetilde{V}^y_{ij;1} \; \widetilde{N}^z_{yx} = \widetilde{V}^z_{ij;x} \tag{19.5}$$

More explicitly (19.3) reads

$$\begin{aligned}
\widetilde{N}_{yx}{}^{z} &= \sum_{j,\bar{j};\alpha,\beta,\gamma} \Psi_y^{(j,\bar{j};\alpha,\beta)} \frac{\Psi_x^{(j,\bar{j};\beta,\gamma)}}{\sqrt{S_{1j}S_{1\bar{j}}}} \Psi_z^{(j,\bar{j};\alpha,\gamma)*} , \\
\widetilde{N}_x \widetilde{N}_y &= \sum_z \widetilde{N}_{xy}{}^{z} \widetilde{N}_z .
\end{aligned} \tag{19.6}$$

From (18.20) it follows that

$$\widetilde{N}_{yx}{}^{1} = \delta_{xy^*} , \tag{19.7}$$

This algebra is associative, but in general non-commutative - whenever the corresponding modular invariant matrix $Z_{j\bar{j}}$ has entries larger than 1, like e.g. in the $\widehat{sl}(2)$ $D_{2\ell}$ cases. If all $Z_{ij} = 1$, the summation over α, β, γ in (19.6) is trivial, and this equation is just the spectral decomposition of the matrices $\widetilde{N}$ in terms of the one-dimensional representations Ψ_x/Ψ_1 of the algebra. If, however, some $Z_{ij} > 1$, the matrices $\widetilde{N}$ are not simultaneously diagonalisable, but rather *block-diagonalisable* with blocks $\gamma_x^{(J;\beta\,\gamma)} = \Psi_x^{(J;\beta,\gamma)}/\Psi_1^{(J)}$ realising $Z_{j\bar{j}}$ - dimensional representations $\tau_J(\widetilde{N}_x) = \gamma_x^J$

$$\sum_\beta \gamma_y^{(J;\alpha\,\beta)} \gamma_x^{(J;\beta\,\gamma)} = \sum_z \widetilde{N}_{yx}{}^{z} \gamma_z^{(J;\alpha\,\gamma)} . \tag{19.8}$$

The $\widetilde{N}$ decomposes as

$$\widetilde{N}_x = \sum_{J,\alpha} \sum_{\beta,\gamma} \gamma_x^{J;\beta\,\gamma} \mathbf{e}_{\beta\,\gamma}^{J;\alpha} = \sum_J \gamma_x^J \sum_{\alpha=1}^{Z_{j\bar{j}}} \mathbf{e}^{J;\alpha} , \tag{19.9}$$

where $\mathbf{e}_{yz;\beta\,\gamma}^{J;\alpha} = \Psi_y^{(J;\alpha,\beta)} \Psi_z^{(J;\alpha,\gamma)\,*}$,

$$(\mathbf{e}_{\beta\,\gamma}^{J;\alpha} \mathbf{e}_{\beta'\,\gamma'}^{K;\alpha'})_{xz} = \sum_y \mathbf{e}_{xy;\beta\,\gamma}^{J;\alpha} \mathbf{e}_{yz;\beta'\,\gamma'}^{K;\alpha'} = \delta_{JK}\, \delta_{\gamma\beta'}\, \delta_{\alpha,\alpha'} \mathbf{e}_{xz;\beta\,\gamma'}^{J;\alpha} , \quad \sum_{J,\alpha,\beta} \mathbf{e}_{xy;\beta\,\beta}^{J;\alpha} = \delta_{x,y} \widetilde{N}_1 \tag{19.10}$$

The formula (19.9) is then interpreted as a decomposition of the regular representation of the $\widetilde{N}$-algebra into a sum of representations τ_J, each appearing with multiplicity $Z_{j\bar{j}}$ so the dimension is $\sum_{j\bar{j}} Z_{j\bar{j}}\ \dim(\tau_J) = \sum_{j\bar{j}} Z_{j\bar{j}}^2 = |\widetilde{\mathcal{V}}|$. The representations are in one-to-one correspondence with the physical spectrum $(j, \bar{j};\, \alpha)$ of the bulk theory.

The noncommutativity of $\widetilde{N}$ modifies the analogue of the symmetry relations (15.17) according to

$$\widetilde{N}_{yx}{}^{z} = \widetilde{N}_{x^* y^*}{}^{z^*} = \widetilde{N}_{zx^*}{}^{y} . \tag{19.11}$$

Analogously to a similar property of the NIM-reps and the graph G algebras $\hat{N}$, the $\widetilde{N}$-algebra provides a module of the double fusion algebra - (19.5) is a special case,

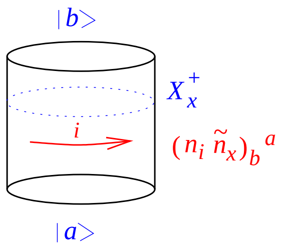
$$\widetilde{V}_{ij}\widetilde{N}_x = \widetilde{N}_x\widetilde{V}_{ij} = \sum_y \widetilde{V}_{ij;x}{}^y\,\widetilde{N}_y\,.$$

This equation can be taken as determining $\widetilde{N}$ given $\widetilde{V}$, with the requirement that the entries of the matrix $\widetilde{N}$ should be non-negative integers. compare with the graph algebra $\hat{N}$

$$n_i\hat{N}_a = \sum_b n_{i1}{}^b\,\hat{N}_b\,.$$

$\widetilde{N}$ - graph algebra associated with Ocneanu graphs. Both types of eqns are solved along with the NIM-reps equations by the algorithm of Xu [79].

19.2. Defects on the cylinder

- Similarly on the cylinder 

one can consider both boundaries and defect lines. Using that

$$P^{(j,\bar{j};\alpha,\beta)}|a\rangle = \delta_{j\bar{j}}\,\psi_a^{(j,\beta)}|j,\alpha\rangle\!\rangle$$

hence

$$\begin{aligned} X_x|a> &= \sum_{j,\alpha}\sum_{\beta} \frac{\Psi_x^{j,j;\alpha,\beta}}{S_{1j}}\,\frac{\psi_a^{j,\beta}}{\sqrt{S_{j1}}}\,|j,\alpha>> \\ &= \sum_c \tilde{n}_{ax}{}^c|c> \end{aligned} \tag{19.12}$$

where

$$\tilde{n}_{ax}{}^c = \sum_{j,\alpha,\beta} \psi_a^{(j,\alpha)}\,\frac{\Psi_x^{(j,j;\alpha,\beta)}}{\sqrt{S_{1j}S_{1\bar{j}}}}\psi_c^{(j,\beta)*}\,. \tag{19.13}$$

$(j,\alpha),(j,\beta)\in\mathcal{E}$ and $\psi_a^{(j,\alpha)}$ is the diagonalising matrix in (15.16). Thus when considered on the cylinder, the defects can be interpreted as operators that map conformal boundary conditions into conformal boundary conditions. Repeating the derivation of Cardy equation one finds that on a cylinder with one defect line X_x and boundary states a and b, a new set of non-negative integer multiplicities occurs in the decomposition of

$$\mathcal{H}_{b|x\,a} = \oplus_{i\in\mathcal{I}}(\tilde{n}_x\,n_i)_a{}^b\;\mathcal{V}_i = \oplus_{i\in\mathcal{I}}(n_i\tilde{n}_x)_a{}^b\;\mathcal{V}_i$$

• The matrices $\tilde{n}_x$ do not commute for nontrivial $Z_{jj} > 1$ and form a NIM-rep of the $\widetilde{G}$-graph algebra $\widetilde{N}$

$$\tilde{n}_x \tilde{n}_y = \sum_z \widetilde{N}_{xy}{}^z \tilde{n}_z \,. \tag{19.14}$$

Since $\tilde{n}_{ax}{}^b$ are interpreted as multiplicities both (19.13) and (19.6) have to give non-negative integers, i.e., one has to find a basis Ψ, along with ψ, such that the integrality of the $\tilde{n}\,,\widetilde{N}$ holds true. All these data were computed for the $\widehat{sl}(2)$ modular invariants. The $D_{\rm even}$ series provides the simplest example of non-commutative algebra $\widetilde{N}$.

• The multiplicities $\tilde{n}$ and $\widetilde{N}$ determine the dual spaces with states corresponding to the last two types of triangles, which enter the construction of the dual WHA $\hat{\mathcal{A}}$.

- Examples of WZW defects

Diagonal defects, γ - h.w. of integrable representation of KM algebra

$$X_\gamma = \sum_\mu \frac{S_{\gamma\mu}}{S_{\mathbf{1}\mu}} P^{(\mu,\mu)}$$

Explicit realisation - by Wilson loop operators, e.g. [Bachas, Gaberdiel (2004)]

$$\mathcal{O}(\lambda; R) = \mathrm{Tr}_R P \, \exp(i\lambda \oint_C dx^+ J^a t^a) \, , \ \lambda = \lambda^* = -\frac{1}{k}$$

Currents

$$J(x^+) = -ik\partial_+ g g^{-1} \, , \ \bar{J}(x^-) = ikg^{-1}\partial_- g \, , \ g(x^+, x^-) \in G$$

generate symmetry of the WZW model

$$g \to u(x^+)^{-1} \, g \, \bar{u}(x^-)$$

and transform as (gauge fields)

$$J \to u^{-1} J u + ik \, u^{-1} \partial_+ u$$

These observables - measure $Tr_R(M)$ the monodromy of the classical solutions of the WZW eqs;
Quantum operators - need regularisation - analysed perturbatively up to 4-th order in the powers of λ

$k \to k + h^\vee$

Eigenvalues in $\mathcal{H}_\mu \otimes \bar{\mathcal{H}}_\mu$

$$\mathcal{O}(\lambda^*; R_\gamma) = \frac{S_{\gamma\mu}}{S_{\mathbf{1}\mu}}$$

Non-perturbative quantisation - [Alekseev, Monier (2007)] :

Wilson loops identified with central elements

in (completions) of the universal envelopping algebra $U(\mathfrak{g})$ of the affine Kac-Moody algebra $\mathfrak{g}$; - constructed by [Kac (1984)].

In condensed matter physics such operators encountered - in Kondo problem - magnetic impurities in (critical) bulk metal; boundary perturbations - RG flows relating conformal boundary conditions.

19.3. Generalised Pasquier algebra and the OPE coefficients

The last set of matrices that we may associate with the Ocneanu graph generalises the Pasquier algebra.

We can define a dual of the $\widetilde{N}$ algebra by the algebra of linear maps

$$\begin{aligned}
\triangle^+_{J;\beta,\beta'} \, : \; \widetilde{N}_x \to \triangle^+_{J;\beta,\beta'}(\tilde{N}_x) &= \frac{\Psi_x^{(J;\beta,\beta')}}{\Psi_x^1} \,, \\
(\triangle^+_{I;\alpha,\alpha'} \, \triangle^+_{J;\beta,\beta'})(\widetilde{N}_x) &= \triangle^+_{I;\alpha,\alpha'}(\tilde{N}_x) \, \triangle^+_{J;\beta,\beta'}(\tilde{N}_x) \\
&= \sum_{K;\gamma,\gamma'} \widetilde{M}_{(I;\alpha,\alpha')\,(J;\beta,\beta')}{}^{(K;\gamma,\gamma')} \; \triangle^+_{K;\gamma,\gamma'}(\tilde{N}_x)
\end{aligned} \tag{19.15}$$

with structure constants

$$\begin{aligned}
\widetilde{M}_{(I;\alpha,\alpha')\,(J;\beta,\beta')}{}^{(K;\gamma,\gamma')} &= \sum_x \frac{\Psi_x^{(I;\alpha,\alpha')}}{\Psi_x^{(1)}} \, \Psi_x^{(J;\beta,\beta')} \, \Psi_x^{(K;\gamma,\gamma')\,*} \,, \\
\sum_{k,\bar{k},\gamma,\gamma'} \widetilde{M}_{(I;\alpha,\alpha')\,(J;\beta,\beta')}{}^{(K;\gamma,\gamma')} \frac{\Psi_x^{(K;\gamma,\gamma')}}{\Psi_x^{(1)}} &= \frac{\Psi_x^{(I;\alpha,\alpha')}}{\Psi_x^{(1)}} \, \frac{\Psi_x^{(J;\beta,\beta')}}{\Psi_x^{(1)}}
\end{aligned} \tag{19.16}$$

This "generalised Pasquier algebra" is abelian. We shall demonstrate that it plays the same role as the Pasquier algebra in determining, now the general, spin field, relative OPE coefficients.

• We shall consider 4-point functions of physical fields in the presence of defects. It is sufficient to look at the functions on the plane, which can be interpreted as the $L/T \to \infty$ limit of the torus correlators, $\lim_{L/T\to\infty} \mathrm{Tr}(e^{-2LH} \, ...)$, when we map it to the plane through $w \to z = ^{\frac{-2\pi i w}{T}}$. Let us sketch the argument which is a generalisation of the derivation of the locality equations.

Consider a 4-point function with insertion of two twist operators (18.5), here $J = (j, \bar{j})$ and the dependence of the CVO multiplicity indices is suppressed

$$\begin{aligned}
&\langle 0| \Phi_{(J^*;\beta)}(z_1, \bar{z}_1) \, \Phi_{(I^*;\alpha)}(z_2, \bar{z}_2) \, X_x \, \Phi_{(I;\alpha')}(z_3, \bar{z}_3) \, \Phi_{(J;\beta')}(z_4, \bar{z}_4) \, X_x^\dagger |0\rangle \\
&= \sum_{k,\bar{k},\gamma,\gamma'} d^{(1)}_{(J^*;\beta)(J;\beta^*)} \; d^{(J;\beta^*)}_{(I^*;\alpha)(K;\gamma)} \; \frac{\Psi_x^{(k,\bar{k};\gamma,\gamma')}}{\Psi_1^{(k,\bar{k})}} \; d^{(K;\gamma')}_{(I;\alpha')(J;\beta')} \; \frac{\Psi_x^{(1)}}{\Psi_1^{(1)}} \\
&\quad \langle 0| \phi^1_{j^*j}(z_1) \, \phi^j_{i^*k}(z_2) \, \phi^k_{ij}(z_3) \, \phi^j_{j1}(z_4)|0\rangle \times (\text{right chiral block}) \,,
\end{aligned} \tag{19.17}$$

taking into account that $d^{(J;\beta')}_{(J;\beta')(1)} = 1$. To give meaning of (19.17) we have assumed that the decomposition of the physical fields involves several copies of each product of left and right chiral blocks,

$$\Phi_{I;\alpha}(z, \bar{z}) = \sum_{j,\bar{j},k,\bar{k},\beta,\gamma,t,\bar{t}} d^{(K;\gamma);t,\bar{t}}_{(I;\alpha)(J;\beta)} \; \Big(\phi^k_{ij;t}(z) \otimes \phi^{\bar{k}}_{\bar{i}\bar{j};\bar{t}}(\bar{z})\Big)^{\gamma}_{\alpha\,\beta} \tag{19.18}$$

as usually the summation over the descendants is suppressed. We have also used that

$$\langle 0|\Phi_{(J^*,\alpha)}X_x\Phi_{(J',\beta)}|0\rangle = \delta_{j,j'}\delta_{\bar{j},\bar{j}'}\frac{\Psi_x^{(J;\alpha,\beta)}}{\Psi_1^J}\langle 0|\Phi_{(J^*,\alpha)}\Phi_{(J,\beta)}|0\rangle \tag{19.19}$$

The limit $z_{21}, z_{34} \to \infty$ of the correlator (19.17) is alternatively represented by the identity contribution in the correlator

$$\begin{aligned}
&\langle 0|\Phi_{(I^*;\alpha)}(z_2,\bar{z}_2)\ X_x\,\Phi_{(I;\alpha')}(z_3,\bar{z}_3)\,\Phi_{(J;\beta')}(z_4,\bar{z}_4)\,X_x^\dagger\,\Phi_{(J^*;\beta)}(z_1,\bar{z}_1)|0\rangle \\
&= d^{(1)}_{(I^*;\alpha)(I;\alpha^*)}\ \frac{\Psi_x^{(i,\bar{i};\alpha,\alpha')}}{\Psi_1^{(i,\bar{i})}}\ d^{(1)}_{(J;\beta')(J^*;\beta'^*)}\ \frac{\Psi_x^{(j,\bar{j};\beta,\beta')}}{\Psi_1^{(j,\bar{j})}} \\
&\quad \langle 0|\phi^1_{i^*i}(z_2)\,\phi^i_{i1}(z_3)\,\phi^1_{jj^*}(z_4)\,\phi^{j^*}_{j^*1}(z_1)|0\rangle \times (\text{right chiral block}) + \dots\,,
\end{aligned} \tag{19.20}$$

using (19.19) again. Next, as in the proof of the crossing relations, we use the braiding relations for the chiral blocks to identify the two products of chiral correlators , i.e., move j^* and $\bar{j}^*$ to the very right– this brings about the product of fusing matrices $F_{k1}\begin{bmatrix} j^* & j \\ i & i \end{bmatrix}\ F_{\bar{k}1}\begin{bmatrix} \bar{j}^* & \bar{j} \\ \bar{i} & \bar{i} \end{bmatrix}$. In a unitary gauge they are given by square roots of q-dimensions, which precisely match the factors Ψ_1.

The final result reads (restoring the dependence on the CVO multiplicity indices)

$$\frac{\Psi_x^{(I;\alpha,\alpha')}}{\Psi_x^{(1)}}\ \frac{\Psi_x^{(J;\beta,\beta')}}{\Psi_x^{(1)}} = \sum_{k,\bar{k},\gamma,\gamma'}\ \sum_{t,\bar{t}} d^{(K^*;\gamma;t,\bar{t})}_{(I^*;\alpha)(J^*;\beta)}\ d^{(K;\gamma';t,\bar{t})}_{(I;\alpha')(J;\beta')}\ \frac{\Psi_x^{(K;\gamma,\gamma')}}{\Psi_x^{(1)}} \tag{19.21}$$

Comparing with (19.16) we see that we can identify the product of OPE coefficients in the r.h.s. of (19.21) with the matrix elements of M. In particular, for coinciding $\gamma = \gamma'$ we have

$$\sum_{t,\bar{t}} |d^{(K;\,\gamma);t,\bar{t}}_{(I;\,\alpha)(J;\,\beta)}|^2 = \widetilde{M}_{(I;\,\alpha,\alpha)\,(J;\,\beta,\beta)}{}^{(K;\,\gamma,\gamma)} \tag{19.22}$$

i.e., the modulus square of the relative structure constants of the OPA of the corresponding CFT are expressed in terms of the (diagonal elements of the) structure constants (19.16) of the generalised Pasquier algebra.

• This formula is checked in the $\widehat{sl}(2)$ cases to confirm all the numerical data obtained by solving the locality equations (6.9). In the cases related to conformal embeddings the formula reproduces the factorisation of the spin field constants into matrix elements of Pasquier algebra. In the case D_{even}, the only one with multiplicities, there is a weaker form of this factorisation, which is also reproduced. Most impressively, the formula (19.21) confirms the OPE spin constants of the exceptional case E_7, in which there is no such left-right factorisation. The (squares of the) OPE coefficients can be now computed with (19.22) explicitly in higher rank cases like $\hat{sl}(3)$ or $\hat{sl}(4)$ modular invariants for which the Ocneanu graphs have been constructed.

• In this computation we have used the defects without specifying them. On the other hand we can consider "twisted" correlators inserting different X_x and taking specific choices for x. This has sense in the cases when the defects correspond to a group like in the $\mathbb{Z}_2$ related twisted partition functions of the diagonal $sl(2)$ cases. In this way one may reproduce the order-disorder parameter correlators, with half-spin OPE content, in general parafermions, etc. effectively assigning the defect to one of the local operators.

19.4. Comments

• Based on these constructions the Ocneanu "double triangle algebra" (a "weak C^*-Hopf algebra") can be interpreted as a novel quantum symmetry, intrinsic to any rational 2d conformal field theory with a Verlinde type formula (10.6). The Ocneanu cells ${}^{(1)}F$, the $3j$-symbols of the weak Hopf algebra, reappear in several guises.

• The definition (17.14) of the generalised CVO is analogous to the definition of the operators covariant with respect to the action of the quantum group in which the finite dimensional representations of the quantum groups $U_q(\overline{\mathfrak{g}})$ at roots of unity are used as auxiliary spaces. However the OPE of the operators in (17.14) as well as their braiding properties are consistent with the fusion rules in contrast with the properties of their quantum group analogs. The main reason behind this distinction is the different nature of the 3j-symbols, as noted already in [75] for the diagonal cases in which they coincide with the fusing matrices, the 6j symbols - unlike the Clebsch-Gordan coefficients of the quantum groups. Moreover, in this way one defines generalised CVO for *all* non-diagonal modular invariants, corresponding to the matrix representations (15.20). Furthermore the non-uniqueness of the vacuum, usually considered as a deviation from the standard axioms, here in the boundary CFT is justified by the physical interpretation of the set $\mathcal{V}$.

Finally, the Ocneanu quantum symmetry not only contains all the combinatorial data required to define the chiral ingredients of any RCFT, but it also incorporates the full physical spectrum, and contains an information on the relative OPE coefficients of local fields as demonstrated by (19.22).

• The non-local operators defining the Wilson loops can be interpreted as examples of defects [80], [81]. Similar interpretation appears in condensed matter in the study of the Kondo problem [82]. More general defects appear as intertwining operators between two different theories. For further developments see e.g. [83], [84]. In general there are different realisations of the defining relations (18.1), discussed in different contexts.

20. Liouville CFT

Liouville theory is determined by the action

$$A = \int d^2x \big(\frac{1}{4\pi} \partial_\nu \phi(x)\, \partial^\nu \phi(x) + \mu e^{2b\phi(x)} \big) \tag{20.1}$$

Classically it is conformally invariant, up to boundary terms, under logarithmic tranformation law of the free field,

$$z \to z' = z'(z)\,, \qquad \phi(z,\bar{z}) \to \phi'(z',\bar{z}') = \phi(z,\bar{z}) - \frac{1}{2b} \log |\frac{dz'}{dz}|^2$$

such that the added term $dz\, d\bar{z}\, e^{2b\phi(z,\bar{z})}$ to the free field action is invariant. The parameter b is assumed real.

• The quantum Liouville theory can be seen as a conformal theory described by representations of $c > 25$ Virasoro algebra. The free field energy-momentum tensor

$$T(z) = -(\partial_z \phi)^2 + Q \partial^2 \phi\,, \qquad Q = \tfrac{1}{b} + b \tag{20.2}$$

and analogously $\bar{T}(\bar{z})$ leads to a central charge

$$c = 13 + 6(1/b^2 + b^2) = 1 + 6(Q)^2$$

Q replaces the parameter α_0, the background charge. The action (20.1) is analogous to the action used in the Coulomb gas representation of the $c < 1$ (minimal) theories with the interaction term interpreted as one of the screening charges, but now $e^{2ib\phi}$ is replaced by $e^{2b\phi}$ with real b and the background charge $i\alpha_0$ is replaced by Q.

• The fields are the vertex operators

$$V_\alpha(z,\bar{z}) = e^{2\alpha\phi(z,\bar{z})}$$

primary with respect to the energy-momentum tensor (20.2) of scaling dimension

$$h(\alpha) = \alpha(Q - \alpha) = \bar{h}(\alpha)$$

• Representations:

The space of states decomposes as

$$\mathcal{H} = \int_{\mathcal{S}} \mathcal{V}_\alpha \otimes \mathcal{V}_\alpha\,, \qquad \mathcal{S} = \tfrac{Q}{2} + i\mathcal{R}^+$$

where $\mathcal{V}_\alpha$ are unitary representations of Vir of h.w. $|v_P\rangle$ with positive real P related to $\alpha = Q/2 + iP$ so that

$$h(\alpha) = h(\tfrac{Q}{2} - P) = \tfrac{Q^2}{4} + P^2$$

It is convenient to work with arbitrary sign of P, i.e., to extend Vir and its representations. The primary states are normalised as

$$\langle v_{P'} | v_P \rangle = \pi \delta(P - P')$$

- For $c > 25$ there are also degenerate series of representations of Vir - non-unitary

$$\beta = -(m-1)\tfrac{b}{2} - (n-1)/2b\,, m, n \in \mathbb{Z}_{\geq 1}, \text{or}, \; \alpha \to Q - \alpha$$

Similarly to the $c < 1$ degenerate series there are differential equations resulting from the factorisation of the singular vectors. Their 3-point OPE coefficients - by analytic continuation from the $c < 1$ ones, just replace $b^2 \to -b^2\, eb \to \alpha b$ (hence $e^2 \to -\alpha^2$). The simplest examples are $\beta = -b/2$ and $\beta = -1/2b$ and the decoupling of the corresponding level 2 singular vectors restricts the n-point functions.

• Formally the correlators are given by the path integral with the action (20.1) and a background charge $-Q$. By a shift $\phi \to \phi - \frac{1}{2b} \log \mu^{-1}$ the dependence on the cosmological parameter μ is reduced to an overall factor

$$\mu^{\frac{1}{b}(Q - \sum_i \alpha_i)}$$

• The path integral is divergent due to the behaviour of the integral over the zero mode ϕ_0 at $-\infty$, $e^{-2bs\phi_0}$, there are poles at $s = (Q - \sum_i \alpha_i)/b = n$ - nonnegative integers. The residue of such pole corresponds to a standard free field Coulomb gas integral with n screening charges of the type in the action (20.1), and a background charge at infinity $-Q$. So any expression for, say the 3-point function should reproduce these singularities.
• The formula of DOZZ does so and in fact it reveals more singularities - as though we have added to the Liouville action the second screening charge $e^{2/b\phi}$ - like dual Liouville action.
• 3-point function (standard coordinate dependence); coefficient:

$$C(\alpha_1\,,\,\alpha_2\,,\,a_3) = \lambda^{\frac{1}{b}(Q - \sum_i \alpha_i)} \frac{\Upsilon_b(b)\,\Upsilon_b(2\alpha_1)\,\Upsilon_b(2\alpha_2)\,\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_{123} - Q)\,\Upsilon_b(\alpha_{23}^1)\,\Upsilon_b(\alpha_{13}^2)\,\Upsilon_b(\alpha_{12}^3)} \tag{20.3}$$

$$\lambda = \mu\,\pi(\gamma(b^2)b^{2-2b^2}$$

- The function

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}$$

is expressed by the Barnes double Gamma function $\Gamma_b(x)$. The function Υ_b is an entire function of x with zeros located at $x = -mb - n/b$ and $Q + mb + n/b$ with $m, n \in \mathbb{Z}_{\geq 0}$. Correspondingly - $\Gamma_b(x)$ a meromorphic function with poles on a lattice of points for $x = -mb - n/b\,,\ m, n \in \mathbb{Z}_{\geq 0}$, generalising the pole structure of the usual Gamma function;

$\Upsilon_b(x)$ satisfies:
functional relations:

$$\Upsilon_b(x + b^{\epsilon}) = b^{\epsilon(1-2b^{\epsilon}x)}\,\gamma(x\,b^{\epsilon})\ \Upsilon_b(x)\,, \qquad \epsilon = \pm 1\,, \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \tag{20.4}$$

self-duality and reflection property

$$\Upsilon_{1/b}(x) = \Upsilon_b(x)\,, \quad \Upsilon_b(x) = \Upsilon_b(Q-x)$$

- Thus the DOZZ expression has poles if the three charges satisfy the charge conservation condition $\alpha_i - Q = -mb - n/b$, or any other obtained by reflection $\alpha_i \to Q - \alpha_i$ of some of the charges. The residue in the first case reproduces the $c > 25$ analog of the expression computed by Dotsenko-Fateev [20], i.e., with $b^2 \to -b^2\,\alpha b \to e\beta$, for generic values of b^2; the case of degenerate representations recovered too.
- Derivation:

  - analytic continuation of the **thermal** analog of the DF formula - using the integral representation for the log of the Gamma function - then the integer number of screening charges - continued to an arbitrary complex number.

  - as solution of a functional relation - a shift by b or $1/b$ as in (20.4). The derivation is the same as the one we discussed in the $c < 1$ case, derived from the locality constraint on a 4-point function with one degenerate field $-b/2$ or $-1/2b$ with three arbitrary fields [18]. These functions are expressed in terms of hypergeometric functions and one derives a functional equation for the 3-point OPE constants of the three arbitrary fields, for which there is no charge conservation requirement. The dependence on the coupling constants $\mu, \tilde{\mu}$ comes from the OPE coefficients of the fundamental field computed with the standard free field representation. The equations and the solutions are invariant under the duality transformation

- duality -

$$b \to 1/b\,, \qquad \lambda \to \tilde{\lambda} := \lambda^{1/b^2}$$

- 2-point function: (do not restrict to $P > 0$ but $P \neq 0$, real)

$$\begin{aligned} C(\alpha_1, \alpha_2) &= 2\pi\delta(\alpha_2 + \alpha_1 - Q) + S(\alpha)\delta(\alpha_2 - \alpha_1) \\ S(\alpha) &= \frac{\lambda^{\frac{1}{b}(Q-2\alpha)}}{b^2} \frac{\gamma((2\alpha - Q)/b)}{\gamma(b(Q-2\alpha)} \\ S(\alpha)\, S(Q-\alpha) &= 1 \end{aligned} \tag{20.5}$$

here S - reflection amplitude

$$C(\alpha_3\,,\, \alpha_2\,,\, a_1) = S(\alpha_3)\, C(Q-\alpha_3\,,\, \alpha_2\,,\, a_1)$$

The identity is beyond the principal series (corresponds to a degenerate field) so it can be reached by a limit, e.g., the limit $\alpha_2 \to 0$ of the 3-point function.

- Observation: The $c < 1$ DF expression also admits an analytic extension for unconstrained charges, which reproduces it directly (no poles, no residues). It is obtained as a solution of the analogous $c < 1$ functional equation and is given by an expression proportional to the inverse of the DOZZ formula

$$C^{(c<1)}(e_1\,,\, e_2\,,\, e_3) = \frac{\lambda_m^{-\frac{1}{b}\left(e_0 - \sum_i e_i\right)}}{\prod_{i=1}^3 b^{\epsilon_i}\, \gamma(b^{\epsilon_i}(Q-2\alpha_i))} \frac{\lambda^{\frac{1}{b}\left(Q - \sum_i \alpha_i\right)}}{C^L(\alpha_1\,,\, \alpha_2\,,\, \alpha_3)} \tag{20.6}$$

if the charges are related as

$$\alpha_i = \epsilon_i e_i + b^{\epsilon_i}, i = 1, 2, 3\,, \qquad \epsilon_i = \pm 1 \tag{20.7}$$

for any choice of the three signs ϵ_i. The overall constant is fixed as in (6.15). This fact plays a role in the non-critical string theory described by two Virasoro theories of central charges summning to 26, and a pair of fermion ghosts of dimensions 2, -1 of central charge -26. The relation of the charges comes from the representation of the tachyon operators (one of the physical fields in the non-critical string) as products of $c < 1$ and $c > 25$ vertex operators of overall dimension 1

$$\triangle^m(e) + \triangle^L(\alpha) = 1$$

with e and α related as in (20.7). In the critical string described by 26 free bosons the Liouville part instead decouples.

- Crossing symmetry of the Liouville correlators: 4-point function, in say, the s channel

$$\langle V_{\alpha_4(\infty)} V_{\alpha_3}(1) v_{\alpha_2}(z) V_{\alpha_1}(0)\rangle = \int d\alpha\, C(\alpha_4, \alpha_3, \alpha) C(Q-\alpha, \alpha_2, \alpha_1) |G(z)|^2$$

where $G(z)$ is the s channel general conformal block, determined in principle by the symmetry, as we discussed.

To prove locality (crossing symmetry) one needs the fusing matrix F for a continious spectrum. It was found by Ponsot and Teschner exploiting the quantum groups $U_q(sl(2,\mathcal{R})$ for $q = e^{2\pi i b^2}$ and $q = e^{2\pi i/b^2}$ and deriving the quantum 6j symbols for a class of representations of both groups. This quantity is given by an integral over ratios of another special function expressed in terms of the Barnes double Gamma function

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$$

satisfying the functional relation

$$S_b(x + b^\epsilon) = 2 \sin b^\epsilon x\, S_b(x)$$

Once again the expression for the fusing matrix develops poles if some triple of charges satisfies a charge conservation condition. By taking residues one recovers in this way the standard (Coulomb gas) fusing matrix - for generic b^2, which can be further continued analytically to reproduce the $c < 1$ case, in particular, the rational values.

- The same quantity - up to an overall gauge depending factor is used in the boundary Liouville theory to obtain the boundary field 3-point functions C. The pentagon eqn is an integral equation.

20.1. Boundary Liouville theory

$$A = \frac{1}{4\pi} \int_{H_+} d^2x \big(\partial_\nu \phi\, \partial^\nu \phi + \mu e^{2b\phi}\big) + \mu_B \int_{\mathcal{R}} dx\, e^{b\phi(x,x)} \tag{20.8}$$

Here μ_B - boundary cosmological constant is considered as a parameter dependent constant, labelling the boundary conditions

$$\cos \pi b(2\sigma - Q) = \lambda_B = \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin b^2 \pi} \tag{20.9}$$

and a dual one with $b \to 1/b$, $\mu_B \to \tilde{\mu}_B$, etc. I.e., in the boundary case the dependence on the cosmological constant is non-trivial.

The boundary terms in the action - screening charges that impose boundary conditions - effective Coulomb gas technique for the computation of the correlators with arbitrary number of bulk and boundary fields - with charge conservation constraints [85].

- There are analogs of the basic boundary structure constants and the equations they satisfy of the rational CFT

- 1-point bulk function

Computed from a 2-point bulk correlator (4-point chiral correlator) with one of the fields chosen to be a fundamental field $V_{-b/2}$. In one of the channels one uses the 2-term OPE expansion of the fundamental field with bulk OPE coefficients known. In the other channel the bulk fields approach the boundary and the fundamental bulk field gives two boundary fields - one extracts the identity contribution.

One gets an analog of the second bulk-boundary Lewellen equation, however it is linear in the 1-point function of the generic operator since the two operators have different spectrum. The corresponding function for the degenerate operator is computed by a bulk-boundary correlator with one boundary screening charge insertion.

One obtains a finite difference eqn

$$U(\alpha|s) = c_+ U(\alpha + b/2|s) + c_- U(\alpha - b/2|s)$$

with coefficients depending on some ordinary Gamma functions. The eqn determines it up to a constant - that one is determined comparing the singularities with the perturbative expansion with respect to the bulk and boundary terms in the action which has sense for values satisfying charge conservation conditions.

For $\alpha = Q/2 + iP$ and the parameter s corresponds to $i(2\sigma - Q)$ if it belongs to the same spectrum

$$U(\alpha|s) = \lambda^{-iP/b}\Gamma(1 + 2ibP))\Gamma(1 + 2iP/b))\frac{\cos 2\pi sP}{iP} =: \frac{\pi\frac{\cos 2\pi sP}{iP}}{W(P)} \tag{20.10}$$

This defines the boundary state $\langle v_P|B_s\rangle$

$$|B_s\rangle = \int_0^\infty dP U(P|s)|P\rangle\rangle = \int_{-\infty}^\infty dP e^{2i\pi Ps} u(P)|P\rangle\rangle$$

It satisfies a reflection property with the Liouville bulk reflection amplitude S

$$U(P) = S(P)U(-P)$$

• The 2-point boundary field function (the non-trivial term in front of $\delta(\beta - \beta'))$ is also found from a finite difference eqn

$$\frac{S(\sigma_1 \pm b/2, \beta + b/2, \sigma_3)}{S(\sigma_1 \pm b/2, \beta, \sigma_3)} = c_{\pm}(\sigma_1, \beta, \sigma_3)$$

(a special case of the pentagon relation) using an insertion of a fundamental field. The coefficients $c_{\pm}(\sigma_1, \beta, \sigma_3)$ are the OPE of a fundamental boundary field and an arbitrary boundary field. They are computed as in the bulk case: the coefficient corresponding to the shift $\alpha \to \alpha - b/2$ is trivial (no screening charges), the other, corresponding to the shift $\alpha \to \alpha + b/2$ - is computed by the insertion of one boundary screening charge - one gets a linear combination of contour integrals, each multiplied by some boundary constant $\cos\pi(2s_i - Q)$ parametrising the contour

$$\begin{aligned} &\int_{-\infty}^{\infty} dw\, \langle\alpha'|e^{2b\phi(w)} V_{-b/2}(x)|\alpha\rangle = -\sum_i \lambda^{1/2} \cos\pi b(2\sigma_i - Q) \oint_{C_i} \dots \\ &= \langle^{\sigma_1} V_{\alpha'}{}^{\sigma \pm b/2}\, V_{-b/2}{}^{\sigma} V_{\alpha}{}^{\sigma_1}\rangle\,, \qquad \alpha + \alpha' - b/2 = Q - b \end{aligned} \tag{20.11}$$

The solution of the functional eqn is expressed by S_b functions

$$\begin{aligned} &S(\sigma_1, \beta_3, \sigma_3) = \\ &\frac{\lambda^{\frac{1}{2b}(Q-2\beta)}\, G_b(2\beta - Q)}{G_b(Q - 2\beta)} \frac{1}{\prod_{s=\pm} S_b(\beta + s(\sigma_2 + \sigma_1 - Q))\, S_b(\beta + s(\sigma_2 - \sigma_1))}\,, \end{aligned} \tag{20.12}$$

This function now serves as a reflection amplitude for the 3-point boundary OPE constant.

$$\begin{aligned} \langle^{\sigma_1} B_{\beta_3}{}^{\sigma_3}\, B_{\beta_2}{}^{\sigma_2} B_{\beta_1}{}^{\sigma_1}\rangle &= {}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \\ &= S(\sigma_1, \beta_3, \sigma_3)\, {}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{Q-\beta_3,\beta_2,\beta_1} \end{aligned} \tag{20.13}$$

• Also the analog of Cardy equation has been derived - with the `NIM`-rep matrix now replaced by a density $\rho(P)$

$$\rho(P) = \int_{\infty}^{\infty} dP\, U(P|s)U(-P|s)e^{-4i\pi PP'}$$

expressed by a derivative of the 2-point boundary function (20.12).

There is a second variant of the boundary Liouville theory [86] in which the spectrum of the boundaries and the boundary fields are described by the discrete set of degenerate Vir $c > 25$ representations. In that case the 1-point function is given by a formula like (20.10) with the same denominator $W(P)$, but with a numerator

$\sim S_{(m,n);\alpha} = \sin \pi bm(2\alpha - Q) \sin \pi \frac{n}{b}(2\alpha - Q)$

(compare with the modular matrix of the rational $c < 1$ case (10.2)). The common denominator has the properties

$$W(P)W(-P) = -S_{(1,1);\alpha}\,,\ W(P)/W(P) = S(P) \tag{20.14}$$

the first relation replacing the square of $\sqrt{S_{1j}}$ in the rational case (15.12).

- Many of the techniques and results were extended to the non-compact WZW models.

Here are some formulae relating the S_b function to q - factorials and common notation used for basic hypergeometric functions (q - hypergeometries)

$$\begin{aligned}
\frac{S_b((m+1)b)}{S_b(b)} &= (2\sin \pi b^2)^m\, [m]_q!\,, \quad S_b(b) = b \\
[m]_q! : &= \prod_{k=1}^{m} \frac{\sin \pi b^2 k}{\sin \pi b^2}\,, \quad q = e^{2\pi i b^2} \\
\frac{S_b(kb+\alpha)}{S_b(\alpha)} &= (-2i\, q^{(k-1)/2}\, e^{\pi i \alpha b})^{-k} \quad (a;q)_k\,, \quad a = e^{2\pi i b \alpha} = q^{\alpha/b} \\
(a;q)_k : &= \prod_{p=0}^{k-1} (1 - a q^p)
\end{aligned} \tag{20.15}$$

Literature: [87], [88], [89], [85], [90], [86], [91].

21. Summary

1. observables - modes of infinite dimensional algebra - Vir , or further extended algebra

2. Hilbert space of states - direct sum of (finite number of) representations , $\mathcal{I}$

3. fields and (chiral) correlation functions

prescribed and organised by the symmetry algebra

4. OPE, fusion $\mathcal{N}$ and duality transformations B,F, S

up to here - *chiral data* : $\mathcal{I}$, $\mathcal{N}$, B, F, S

5. consistency conditions on the coupling of the holomorphic and antiholomorphic chiral parts - on the plane, the torus, on a manifold with a boundary

21.1. Algebra and representations

• For a holomorphic infinitesimal conformal change of coordinates there is a conformal charge associated with the conserved tensor of energy-momentum $T(z)$, $\partial_{\bar{z}} T(z) = 0$

$$Q_\epsilon = \frac{1}{2\pi i} \oint_C dz\, \epsilon(z) T(z)$$

or, in components, the Vir generators L_n - (the moments) of the conserved tensor $T(z)$, provide infinitely many conserved charges - they may be interpreted as observables, like the usual momentum and angular momentum.

• On the plane CFT described by two (complex) Vir algebras, or $T(z), \bar{T}(\bar{z})$ with unrelated $z, \bar{z}$, so that both euclidean and Minkowski space realisations are recovered

The combinations $L_n + \bar{L}_n$ and $i(L_n + \bar{L}_n)$ (for $\bar{z} = z^*$ in the second case) preserve the real 2d euclidean space. In the half-plane we keep only $L_n + \bar{L}_n$.

• Extensions of Vir : e.g. affine KM (current) algebras : the Coulomb gas based on the affine $\hat{u}(1)$; the Sugawara construction for WZW $\widehat{sl}(n)_k$.

The extended algebra may not be a Lie algebra - like the W -algebras (which originates from an affine KM algebra via QHR).

• **Space of states** $\mathcal{H}$ (restricting to one of the algebras, i.e., the chiral sector)

- representation space of the observable (local) algebra, for the rational theories - $\mathcal{H} = \oplus_{i \in \mathcal{I}} \mathcal{V}_i$, where for a fixed value of the central charge the set $\mathcal{I}$ of representations is finite

• (on the plane) each representation appears only once; on a strip (or the half plane) - the representations appear with multiplicity n^b_{ja} depending on the possible boundary conditions

• this set of "superselection sectors" parametrised by $\mathcal{I}$ - contains an identity labelled by $i = 1$ (associated with the envelope of the algebra itself), also an involution $j \to j^*$; the vacuum state of the identity representation - annihilated by part of the modes, including those corresponding to the finite subalgebras;

• these representations - "highest weight" - the eigenvalues of L_0 and $\bar{L}_0$ and hence of $L_0 + \bar{L}_0$ are bounded from below in each irrep i.e., the energy is bounded from below; this interpretation comes from the realisation on the cylinder $z = e^{\frac{2\pi}{L}w}, w = t + iv$, generator of "time" $\partial_t = z\partial_z + \bar{z}\partial_{\bar{z}}$, the eigenvalues of the Hamiltonian $E_0 = \frac{2\pi}{L}(h + \bar{h} - \frac{c}{12})$

In the full space of states the energy is bounded by the lowest value of h_i in the set (the identity in the unitary representations)

• h.w. representations - either Verma modules - we use that these algebras admit a triangular decomposition $n_+ \oplus \mathfrak{h} \oplus n_-$ - the envelope of n_- acts freely;

• irreps - factor representations over (union of) invariant submodules; - generated by singular vectors

• - unitary representations; hermitian form, a hermitian conjugation $X \to X^+$ in the algebra, in some cases positive definite;

NB: We have considered representations s.t. $L_0^+ = L_0$, hence diagonalizable. But there are other "logarithmic" CFT , for which L_0 acts non-diagonally, indecomposable representations.

• some of the local extensions of the algebra - based on the fact that in some cases there are representations characterised by integer conformal dimensions - together with the identity representation (the algebra) - determine the generators of an extended algebra - examples - the conformal embeddings of affine algebras,

Ex: $\widehat{sl}(2)_4 \subset \widehat{sl}(3)_1$, $2j + 1 = 1, 2, 3, 4, 5$, the last representation has Sugawara dimension $= 1$

characters - generating functions of the multiplicities of states in the module;

21.2. Chiral fields and correlators; duality transformations

• **Fields** - operators in the Hilbert space;

• themselves realise the irreps - state- field correspondence;

• each representation - a familty of fields, the the h.w. state - primary field; descendants defined via radially ordered OPE expansions with $T(z)$ or the currents J;
in terms of correlators - Ward identities ; reduce the correlators of descendant fields to differential operators acting on correlators of primary fields

• alternatively - Vir generators act on the primaries as differential operators - and more complicated transformation laws for the descendants;
the Ward identities corresponding to the projective subalgebra - imply in particular that the coordinate dependence of the 2- and 3-point functions fixed (isospin coordinates in the WZW case; all states of the finite dimensional rep of the finite dimensional subalgebra - treated as a primary field)

• not all 3-point functions admissible - possible triples of conformal families - fixed by the infinite symmetry - factorisation of singular (null) vectors -

• **fusion** - the rule which determines which primary, and with what multiplicity, appears in the operator product of two primary fields

$$\Phi_i \star \Phi_j = \sum_k \mathcal{N}_{ij}^k \, \Phi_k$$

• equivalently - chiral vertex operators CVO as interwtining operators, the multiplicity of the space of these operators - fusion multiplicity $\mathcal{N}$; freedom in the normalisation

• fusion (Verlinde) algebra associative, commutative algebra with structure constants $\mathcal{N}$,
ex: $\widehat{sl}(2)$ related models - the $c < 1$ minimal theories (or WZW models) - BPZ fusion rules - truncated rule of decomposition of finite dimensional irreps, directly computed from the factorisation of singular vectors applied to 3-point functions ; in general Verlinde formula for the fusion multiplicities - a spectral decomposition, diagonalising symmetric unitary matrix S

• use OPE with $W(z)$ to rewrite the singular vectors as differential operators -the factorisation condition - equivalent to linear differential equations for the chiral correlators - the conformal blocks - basis of the independent solutions of these differential equations; Examples: the hypergeometric eqn in the Vir case, corresponding to a level 2 singular vector, or the KZ equations in the WZW models

NB: the fields corresponding to the reducible submodule also important and sometimes are not killed - e.g. the logarithimic field ϕ in the Coulomb gas; the irrep corresponds to a constant (operator), in the kernel of the derivative , so that the singular vector $L_{-1}|0\rangle$ is factorised. But $\partial_z \phi \neq 0$ and gives the current $J(z)$;
another example - used in Liouville theory applied to non-critical string: the level 2 singular vector - differential eqn - applied to $V^L_{-b/2}$ gives zero, but applied to $\phi V^L_{-b/2}$ - produces $V^L_{3b/2}$ - dressing field of the $c < 1$ vertex operator $V_{b/2}$. [92].

Duality transformations

- on the plane the chiral blocks can be obtained by "glueing" 3-point functions in different ways - different bases; or, chiral blocks - multivalued functions, the analytic continuation relates different sheets of the finite (due to rationality) covering of the Riemann sphere - braiding B and fusing F of CVO - numerical (complex valued) linear transformations
- or, equivalently, for the operator product of chiral vertex operators - associativity - polynomial equations - pentagon, hexagon
- on the torus - characters (originally defined as generating functions of the multiplicities of states in the module), when considered as functions on the torus parameter τ carry a finite dimensional unitary rep of the modular group $sl(2,\mathbb{Z})$; generators S, T - symmetric functions; more generally $S(j)$ for 1-point function, and for 2-point functions; a further relation for the set of duality tranformations S, F, B. - implies Verlinde formula - spectral decomposition for the multiplicity matrix $\mathcal{N}_j$ with the unitary modular matrix S - as the diagonalising matrix.

21.3. Physical spectrum and correlators - consistency constraints

• Modular invariance

$$\mathcal{H}_P = \oplus Z_{j\bar{j}} \mathcal{V}_j \otimes \overline{\mathcal{V}_{\bar{j}}} \ , \tag{21.1}$$

determines the admissible spectrum of physical fields , i.e., the multiplicity of possible pairs of left-right reps , s. that the identity rep appears once $Z_{00} = 1$ - determined by the modular invariance

• the correlators on the plane - holomorphic factorisation

$$G_4(z, \bar{z}) = \sum_{j,\bar{j}} a_{j,\bar{j}} G_j(z) G_j(\bar{z})$$

• for the correlators - composition of braiding of the holomorphic and anti-holomorphic blocks yields monodromy invariant correlators;

• (euclidean) locality - symmetry- expressed as different chiral decomposition of the local correlators - crossing symmetry

• imposes constraints on the 3-point constants (OPE coeffs) of the physical fields. equivalently - physical fields

$$\Phi_{j,\bar{j}}(z, \bar{z}) = \sum_{f,\bar{f},k,\bar{k}} d^F_{JK} V^f_{j\,k}(z) \otimes \overline{V}^{\bar{f}}_{(\bar{a}\bar{k}}(\bar{z})$$

• diagonal solution (scalars) always exists - diagonal Z, since S and F - unitary

but there are more solutions describing fields with non-trivial integer spin, and the classification of modular invariants should exhaust all the possibilities

• on manifolds with boundary - relation between open and closed sectors - spectrum depends on the boundary conditions; use modular covariance to relate them

• equivalent representations for the partition function - new NIMreps of the Verlinde algebra;

• analogously, alternative representations for the correlators of bulk and boundary fields - determine the structure constants of bulk-boundary OPE algebra, ${}^{(1)}F$, and R, determine also the relative scalar OPE coefficients;

• topological conformal interfaces (defects)

- weakened the modular invariance, allowing topological operators commuting with both algebras, to deform the periodicity condition leading to the torus

• - interpreted as part of the combinatorial data in the definition of the Ocneanu quantum algebra; extended to correlators - gives info on the general OPE coefficients

• relation to Ocneanu quantum symmetry - allows to define and extend the boundary fields as generalised CVO, new braiding $\hat{B}$ (connection to the Boltzmann weights of lattice ADE models) etc.

21.4. Topics not covered

• CFT on higher genus Riemann surfaces; KZB eqs

• logarithmic CFT

• orbifold CFT

• free fermion, supersymmetric algebras and representations, parafermionic algebras and representations

• the role of Galois symmetry in CFT, in particular in the classification of modular invariant problem

• geometric interpretation of the boundary conditions

etc, etc,...

• this was more or less a traditional exposition of the CFT -"conformal bootstrap"; there are alternative and more rigorous approaches like the vertex operator algebras, or the C^* algebraic approach , categories, subfactors....

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