ELEMENTARY REPRESENTATIONS AND INTERTWINEING OPERATORS FOR THE GROUP SU*(4)

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Global realizations of all elementary induced representations (EIR) of the group SU*(4), which is the double covering group of SO^*(5, 1), are given. The Knapp-Stein intertwining operators are constructed and their harmonic analysis carried out. The invariant subspaces of the reducible EIR are introduced and the differential intertwining operators between partially equivalent EIR are defined. Invariant sesquilinear forms on pairs of invariant subspaces are constructed. Differential identities between invariant sesquilinear forms on pairs of irreducible components of the reducible representations are derived. The results will be applied elsewhere to the nonperturbative analysis of Euclidean conformal invariant quantum field theory with fields of arbitrary spin.

Introduction

The group SU*(4) is the double covering group of SO^*(5, 1) (the identity component of O(5, 1)). These groups and their representation theory are studied because the group O^*(5, 1) (including reflection of the first five axes) is the Euclidean conformal group for 4 space-time dimensions. In a previous work [4] the so-called type I (or symmetric tensor) representations of O^*(N, 1) were studied in detail. The reader can find there also a rather comprehensive review of results and methods useful in considering the representation theory of semi-simple Lie groups (especially of split-rank one).

The aim of this paper is to give a global construction of all elementary representations of SU*(4) (and thus of all one- and two-valued representations of SO^*(5, 1)) and to construct and study the intertwining operators between (partially) equivalent representations. We follow the general framework of ref. [4] and whenever possible cast the results in such a form that comparison with the case SO^*(5, 1) (as considered in [4]) is straightforward. The present paper will make possible the extension of the applications (see [5]) to conformal invariant quantum field theory, e.g. we shall be able to consider fields of arbitrary spin.

We define the group SU*(4) as in [10], where also one can find some results about
the group structure. We define an appropriate Cartan involution in the Lie algebra $su^*(4)$ and construct the Iwasawa decompositions for $su^*(4)$ and the Iwasawa and Bruhat decompositions for $SU^*(4)$ following the general scheme for semisimple groups (cf. [7, 13, 2, 18]) and give the relation between these two decompositions. We also display the isomorphism between $su^*(4)$ and $so(5, 1)$.

We define the unitary irreducible representations (UIR) of $M = SU(2) \times SU(2)$ (double covering of $SO(4)$) as in [1]. We display two standard realizations of the elementary induced representations (EIR) following [14, 18] and construct the infinitesimal generators for them.

It is a well known result (cf. [12, 16]) that almost all EIR are irreducible. We construct explicitly the Knapp–Stein intertwining operators [14] and carry out their harmonic analysis.

We also construct invariant sesquilinear forms on pairs of EIR of two types (the second involving the integral kernel of the Knapp–Stein intertwining operators). We obtain the result (which could also be derived from more general considerations) that the complementary series of unitary representations occur only for symmetric tensor representations of $M$ (or $SO(4)$). For that reason Grensing [10] has constructed the Knapp–Stein intertwining operators only for such representations.

We study in detail the exceptional EIR. We construct the invariant subspaces of the reducible representations and define explicitly the differential intertwining operators some of which were introduced for type I representations of $SO^+(5, 1)$ in [4]. Such operators for arbitrary representations of $SO^+(N, 1)$ were also studied in [6]. We discuss their results in Subsection 5.B. After the present paper was completed we received a paper by Zhelobenko [22]. The author has shown that for any real reductive group the intertwining operators between reducible representations (besides the Knapp–Stein operators), correspond to the positive noncompact roots of the root system of $g^C$ relative to $h^C$ ($h^C$ is the complexification of the Cartan subalgebra $h$). The list of these intertwining operators for all simple real algebras of split rank 1 is given. The operators are defined infinitesimally in a realization of the elementary representations with a representation space consisting of complex-valued functions on the group.

We also prove irreducibility of two series of representations partially equivalent to reducible representations. We derive some differential identities between invariant sesquilinear forms on pairs of irreducible components of exceptional representations.

We note that the formulae for the infinitesimal generators of the representations of $M$ (and $SO(4)$) and $G$ (and $SO^+(5, 1)$ (and some results derived by infinitesimal methods) look similar to analogous formulae for $SO^+(3, 1)$ (see [17]) and $SO_0(4, 2)$ (see [20]) respectively. This happens because the representations of $SO^+(3, 1)$ are also realized in the space of polynomials of two complex (Pauli) spinors and the Lie algebras of $SO(4)$ and $SO^+(3, 1)$ are connected through the Weyl unitary trick. However, the results for the representations can not be compared directly because the corresponding groups differ significantly (cf. [15]).
1. Group structure

1.A. The group SU*(4). First we introduce some notation. Let
\[ q_k = -i\sigma_k \quad (k = 1, 2, 3), \quad q_4 = 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \] (1.1)
where \(\sigma_k\) are the Pauli matrices
\[ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

It is well known that the matrices \(q_\mu (\mu = 1, 2, 3, 4)\) are a representation of the quaternion units satisfying
\[ q_4 q_\mu = q_\mu q_4 = q_\mu, \quad q_j q_k = \delta_{jk} - \epsilon_{jkl} q_l. \] (1.2)

An arbitrary (real) quaternion is parametrized by 4 real numbers \(\alpha_\mu\) and is written as
\[ q = \alpha_\mu q_\mu = \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_4 \in \mathbb{Q} \] (1.3)
(where \(\mathbb{Q}\) stands for the quaternion field).

Then by definition \(U^*(4)\) is a matrix group, whose elements are described by
\[ U^*(4) \equiv \left\{ g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha_\mu, \beta_\mu, \gamma_\mu, \delta_\mu \in \mathbb{R}, \quad \det g \neq 0 \right\}. \] (1.4)

The 16-parameter group \(U^*(4)\) is thus isomorphic to \(GL(2, \mathbb{Q})\).

Now we can define the matrix group \(G\):
\[ G = SU^*(4) = \{ g \in U^*(4), \det g = 1 \}. \] (1.5)

In the standard definition of \(SU^*(4)\) [13] the matrices differ from (1.5) by a real orthogonal transformation. The present definition was introduced in [10] without pointing out the connection to the quaternions.

By a straightforward computation we obtain
\[ \det g = \alpha^2 \delta^2 + \beta^2 \gamma^2 - 2(\alpha \cdot \beta)(\gamma \cdot \delta) - 2(\alpha \cdot \gamma)(\beta \cdot \delta) + 2(\alpha \cdot \delta)(\beta \cdot \gamma) + 2\epsilon_{\mu
\nu\lambda\sigma} \alpha_\lambda \beta_\mu \gamma_\nu \delta_\sigma, \] (1.6a)

where
\[ \alpha^2 \equiv \alpha_\mu \alpha_\mu = \alpha^\dagger \alpha = \det \alpha, \quad \alpha \cdot \beta \equiv \alpha_\mu \beta_\mu = \frac{1}{2} (\alpha \beta^\dagger + \beta \alpha^\dagger), \] (1.6b)
\[ \epsilon_{1234} = 1, \quad \alpha^\dagger \equiv \alpha_\mu q_\mu^\dagger, \]
\(q_\mu^\dagger\) is the hermitian conjugated matrix \((q_\mu^\dagger = q_\mu, q_j^\dagger = -q_j)\). Thus condition \(\det g = 1\) gives one constraint and the group \(SU^*(4)\) is 15-dimensional (like the group \(O(5, 1)\)).

It is useful to have different expressions for \(\det g\), namely
\[ \det g = \alpha^2 \delta^2 + \beta^2 \gamma^2 - \alpha \gamma^\dagger \delta \beta^\dagger + (\alpha \gamma^\dagger \delta \beta^\dagger)^+, \] (1.6c)
(where \((\gamma\gamma^+\beta\beta^+)^+ = \beta\delta^+\gamma^+\)) and three more expressions obtained from (1.6c) by simultaneous cyclic permutation of the matrices in the last two terms.

1.B. The Lie algebra of SU*(4). It is easily seen that the Lie algebra of \(G\) is

\[ \mathfrak{g} = \left\{ Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R}, \, \text{tr} Z = 0 \right\} \equiv \mathfrak{su}^*(4). \]  

(1.7)

Noting that \(\text{tr} Z = \text{tr} a + \text{tr} d = 2(a_4 + d_4)\) we shall give an explicit expression for the basis of the 15-dimensional real vector space \(\mathfrak{g}\)

\[ X_k = \frac{1}{2} \begin{bmatrix} q_k & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_k = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & q_k \end{bmatrix}, \quad S_\mu = \frac{1}{2} \begin{bmatrix} 0 & q_\mu \\ -q_\mu^* & 0 \end{bmatrix}, \]

\[ D = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad U_\mu = \frac{1}{2} \begin{bmatrix} 0 & q_\mu \\ q_\mu^* & 0 \end{bmatrix}. \]  

(1.8a)

(1.8b)

It is easy to check that the basis elements (1.8a) span the maximal compact subalgebra \(\mathfrak{k}\) of \(\mathfrak{g}\), while the elements (1.8b) span a vector space \(\mathfrak{p}\) [11].

Next we define the Cartan involution \(\theta\)

\[ \theta Z = -Z^+, \quad Z \in \mathfrak{g}. \]  

(1.9)

With this definition we can state

**Proposition 1.1.** The Cartan decomposition of \(\mathfrak{g}\) is given by

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \]  

(1.10)

**Proof:** Having in mind the properties of \(\mathfrak{k}\) and \(\mathfrak{p}\) already mentioned it remains only to prove that

\[ Z \in \mathfrak{k} \Rightarrow \theta Z = Z, \quad Z \in \mathfrak{p} \Rightarrow \theta Z = -Z \]  

(1.11)

which is straightforward application of (1.8).

Next we single out important subalgebras of \(\mathfrak{g}\). Let \(\mathfrak{a}\) be a subspace of \(\mathfrak{p}\) which is maximal subject to the condition \([Z, Z'] = 0\) if \(Z, Z' \in \mathfrak{a}\). It turns out that \(\mathfrak{a}\) is one-dimensional subalgebra of \(\mathfrak{g}\) and we shall take \(D\) of (1.8b) as its generator. The centralizer \(\mathfrak{m}\) of \(\mathfrak{a}\) is spanned by the matrices \(X_k\) and \(Y_k\). It is isomorphic to \(\mathfrak{su}(2) \oplus \mathfrak{su}(2)\).

To construct the Iwasawa decomposition [13] of \(\mathfrak{g}\) we use the restricted root system of \(\mathfrak{g}\) relative to \(\mathfrak{a}\). Let \(\mathfrak{a}^*\) be the space of linear functionals over \(\mathfrak{a}\). Each functional is determined by its value on \(D\). We define for \(\lambda \in \mathfrak{a}^*, \lambda \neq 0\)

\[ \mathfrak{g}_\lambda = \{ Z \in \mathfrak{g} \mid [D, Z] = \lambda(D)Z \}, \]  

(1.12a)

\[ A = \{ \lambda \in \mathfrak{a}^* \mid \lambda \neq 0, \mathfrak{g}_\lambda \neq \{0\} \}, \]  

(1.12b)

We obtain

\[ A = \{ \lambda_+, \lambda_- \}, \quad \lambda_+(D) = \pm 1, \]  

(1.13a)

and the corresponding subalgebras

\[ \mathfrak{n} = \mathfrak{g}_+, \quad \mathfrak{n} = \mathfrak{g}_-. \]  

(1.13b)
are spanned by
\[ T_\mu \equiv S_\mu + U_\mu = \begin{bmatrix} 0 & q_\mu \\ 0 & 0 \end{bmatrix}, \quad C_\mu = U_\mu - S_\mu = \begin{bmatrix} 0 & 0 \\ q_\mu & 0 \end{bmatrix} \] (1.13c)
respectively.

Obviously \( \tilde{n} = 0 \). Using (1.8) and (1.13) we also obtain the decomposition (valid generally for semisimple Lie algebras):
\[ g = \tilde{n} \oplus n \oplus a \oplus \mathfrak{m}. \] (1.14)

It is easily seen that the map
\[ J: \mathfrak{m} \oplus \tilde{n} \rightarrow \mathfrak{k} \]
\[ J(Z + Z') = Z + Z' + \theta Z' \quad \text{(for } Z \in \mathfrak{m}, Z' \in \tilde{n}) \]
is bijective and that
\[ g = \mathfrak{k} \oplus \mathfrak{a} \oplus n \] (1.15)
which is the Iwasawa decomposition of \( g \).

The algebra \( g \) is isomorphic to \( \mathfrak{so}(5,1) \), the isomorphism given explicitly by
\[ X_{jk} = \epsilon_{jkl}(X_i + Y_i), \quad X_{4k} = -X_{k4} = X_k - Y_k, \]
\[ X_{5\mu} = -X_{5\mu} = S_\mu, \]
\[ X_{\mu 0} = -X_{0\mu} = U_\mu, \quad X_{50} = -X_{05} = D. \] (1.16a)

Indeed,
\[ [X_{AB}, X_{CD}] = \eta_{AC}X_{BD} + \eta_{BD}X_{AC} - \eta_{AD}X_{BC} - \eta_{BC}X_{AD} \] (1.17)
where \( A, B, C, D = 0, 1, \ldots, 5, \eta_{11} = \ldots = \eta_{55} = -\eta_{00} = 1, \eta_{AB} = 0 \) for \( A \neq B \).

Note that the subalgebra \( \mathfrak{k} \) is isomorphic to \( \mathfrak{so}(5) \) (and the elements \( X_{\mu \nu} \) span an algebra isomorphic to \( \mathfrak{so}(4) \)).

1.C. Structure of the group \( \text{SU}^*(4) \) and decompositions. The maximal compact subgroup \( K \) of \( G \) is (cf. [11])
\[ K \equiv U^*(4) \cap U(4) = \text{SU}^*(4) \cap \text{SU}(4), \] (1.18)
isomorphic to \( \text{Spin}(5) \). If \( g \in K \)
\[ \alpha \gamma^+ + \beta \delta^+ = 0, \quad \alpha^2 + \beta^2 = 1, \quad \gamma^2 + \delta^2 = 1. \] (1.19a)
It follows that
\[ \alpha^2 = \delta^2, \quad \beta^2 = \gamma^2. \] (1.19b)

We list other important subgroups starting with the abelian noncompact subgroups [10]:
\[ A \equiv \left\{ a \in G \mid a = \begin{bmatrix} \sqrt{|a|} & 1 \\ 0 & \sqrt{|a|}^{-1} \end{bmatrix}, \ |a| \in \mathbb{R}^+ \right\}, \] (1.20)
\[ N = \left\{ n_b \in G \mid n_b = \begin{bmatrix} 1 & 0 \\ b^+ & 1 \end{bmatrix}, \ b \in \mathbb{R}^2 \right\}, \]  
(1.21)

\[ \tilde{N} = \left\{ \tilde{n}_x \in G \mid \tilde{n}_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \ x \in \mathbb{R}^4 \right\}. \]  
(1.22)

Let \( M \) be the centralizer of \( A \) in \( K \). Then

\[ M = \left\{ m \in K \mid m = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \ \det u = \det v = 1 \right\}. \]  
(1.23)

The matrices \( u, v \in SU(2) \), so \( M = SU(2) \times SU(2) \). Let \( M' \) be the normalizer of \( A \) in \( K \).

The Weyl group of the pair \((G, A)\) (the restricted Weyl group) \( W = M'/M \) has two elements: \( W = \{1, \Omega\} \). Let \( \omega \in \Omega \). Then we have (as a matrix \( \omega \in K \))

\[ M' = \{ m' \in K \mid \text{either } m' \in M, \text{ or } m' = \omega m, \ m \in M \}. \]  
(1.24)

The subgroups \( N \) and \( \tilde{N} \) are conjugated under the Weyl transformation \( \tilde{N} = \omega N \omega^{-1} \).

We shall make the following choice for the representative \( \omega \) of the nontrivial element \( \Omega \):

\[ \omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]  
(1.25)

We also note that the (maximal abelian) Cartan subgroup \( H \) of \( G \) is given by

\[ H = AH_M \]  
(1.26a)

where \( H_M \) is a diagonal subgroup of \( M \), i.e. if \( g \in H_M \) then

\[ g = \begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix}, \ u^\prime = \begin{bmatrix} e^{\varphi(\tau)} & 0 \\ 0 & e^{-\varphi(\tau)} \end{bmatrix}, \ \varphi(\tau) \in \mathbb{R}. \]  
(1.26b)

It is easy to check that \( \mathfrak{g}, \mathfrak{a}, \mathfrak{n}, \mathfrak{n}, \mathfrak{m} \) are the Lie algebras of \( K, A, N, \tilde{N}, M \) respectively.

Next we shall construct the Iwasawa decomposition [13] of \( G \). It is well known that every element \( g \in G \) may be represented uniquely in the factorized form \( (k_I, E K, n_b, E N, \beta, E A) \)

\[ g = k_I n_b a_t. \]  
(1.27a)

For the factors in (1.27a) we have:

**Proposition 1.2.** Let \( k_I = \begin{bmatrix} \varphi_I & \beta_I \\ \gamma_I & \delta_I \end{bmatrix} \) and \( g = \begin{bmatrix} \varphi & \beta \\ \gamma & \delta \end{bmatrix} \). Then we have

\[ \varphi_I = \frac{1}{\sqrt{\beta^2 + \delta^2}} (\delta^2 g - \beta^+ \delta^+ \gamma), \ \beta_I = \frac{1}{\sqrt{\beta^2 + \delta^2}} \beta, \]

\[ \gamma_I = \frac{1}{\sqrt{\beta^2 + \delta^2}} (\beta^2 \gamma - \delta^+ \beta^+ \gamma), \ \delta_I = \frac{1}{\sqrt{\beta^2 + \delta^2}} \delta, \]  
(1.27b)

\[ b_I = \beta^+ g + \delta^+ \gamma, \ |a_I| = \frac{1}{\beta^2 + \delta^2}. \]

The proof consists of a straightforward matrix multiplication.
The order of the factors in the Iwasawa decomposition is a matter of convenience (however the expressions (1.27b) depend on it).

Next we proceed with the construction of the (Gel'fand–Naimark)–Bruhat decomposition [2, 71. It is well known that up to a submanifold of lower dimension every element of $G$ can be written in a unique way as a product

$$g = n_{x_b} n_{b_b} a_B m_B$$

(1.28a)

where $n_{x_b} \in \tilde{N}$, $n_{b_b} \in N$, $a_B \in A$, $m_B \in M$.

**Proposition 1.3.** For $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and $\delta \neq 0$ we have

$$x_B = \frac{1}{\delta^2} \beta \delta^+, \quad b_B^+ = \gamma (\delta^2 \alpha^+ - \gamma^+ \delta^+)$$

(1.28b)

$$|a_B| = \frac{1}{\delta^2}, \quad u_B = \frac{1}{\sqrt{\delta^2}} (\delta^2 \alpha - \beta^+ \gamma), \quad v_B = \frac{1}{\sqrt{\delta^2}} \delta.$$

**Proof:** Straightforward matrix multiplication.

**Remark.** Several special cases of (1.28b) (namely (1.32) and (2.21) below) are displayed in [10].

It was mentioned in the introduction that the simply connected group $SU^*(4)$ is a double covering of $SO^1(5,1)$ [11]. The covering is easily displayed using the Iwasawa or Bruhat decompositions. Indeed, let $G = KAN$, $SO^1(5,1) = K_0 A_0 N_0$. Then $A_0$ and $N_0$ are simply connected and isomorphic to $A$ and $N$ (respectively), while $K \cong Spin(5)$ is the double covering of $K_0 = SO(5)$. Analogously if $\tilde{N}A_0 M_0$ and $\tilde{N}_0 N_0 A_0 M_0$ are (locally isomorphic) dense submanifolds of $G$ and $SO^1(5,1)$ respectively, then $\tilde{N}$ and $\tilde{N}_0$ are isomorphic, while $M = SU(2) \times SU(2)$ is the double covering of $M_0 = SO(4)$.

The group $SO^1(5,1)$ is the conformal group of the 4-dimensional Euclidean space $\mathbb{R}^4$ and so we shall consider $G = SU^*(4)$ as the quantum mechanical Euclidean conformal group. First we note that

$$G/NAM \cong K/M \cong \mathbb{R}^4$$

(1.30)
(where $S^4$ is the unit sphere in $R^5$) can be identified with the one-point compactification of $R^4$. The group $G$ acts in a natural way on the homogeneous space $S^4$. This action will be displayed by using the decomposition (1.28) and the isomorphism between $R^4$ and the abelian subgroup $\tilde{N}$. We define as in [4] the transformation $g: x \mapsto x'$ of $R^4$ by
\[ g^{-1}n_x = \tilde{n}_x n^{-1} a^{-1} m^{-1}. \] (1.31)

Thus we obtain
\begin{enumerate}
    \item[(a)] $g = \tilde{n}_x$, translations of $R^4$, \hspace{1cm} $x' = x - x_1$; (1.32a)
    \item[(b)] $g = m$, rotations of $R^4$, \hspace{1cm} $x' = \Lambda^{-1} x$; (1.32b)
    \item[(c)] $g = a$, dilatations of $R^4$, \hspace{1cm} $x' = x/|a|$; (1.32c)
    \item[(d)] $g = n_b$, special conformal transformations of $R^4$; for $1 - 2bx + b^2x^2 \neq 0$
        \[ x' = (x - x_2b)/(1 - 2bx + b^2x^2); \] (1.32d)
\end{enumerate}

for $1 - 2bx + b^2x^2 = 0$, $x' \notin R^4$, $x' = \infty \in S^4$ ($\infty$ is the point added to $R^4$ to compactify it).

1.D. Relationship between the Iwasawa and the Bruhat decompositions.

**Proposition 1.4.** Let us have (for $\delta \neq 0$) $g = k\tilde{n}_b a_1 = n_{x_{a}} n_{b_{a}} a_{b_{a}} m_{b_{a}}$. Then we can express the 2 x 2 matrices (1.27b) in the Iwasawa factors by those of the Bruhat decomposition (1.28b):
\begin{align*}
    \alpha_I &= \frac{1}{\sqrt{1 + x_{a}^2}} u_b, \hspace{1cm} \beta_I = \frac{1}{\sqrt{1 + x_{a}^2}} \tilde{x}_b v_B, \\
    \gamma_I &= -\frac{1}{\sqrt{1 + x_{a}^2}} \tilde{x}_b^+ u_B, \hspace{1cm} \delta_I = \frac{1}{\sqrt{1 + x_{a}^2}} v_B, \\
    \beta_I^+ &= v_B^2 [(x_{a}^+ + (1 + x_{a}^2)\tilde{x}_a^+)] u_B, \hspace{1cm} |a_I| = |a_B|/(1 + x_{a}^2).
\end{align*}

The inverse formulae of (1.33a) are
\[ x_a = -\frac{1}{\alpha_I^2} \alpha_I \gamma_I^+, \hspace{1cm} \tilde{x}_a^+ = (\gamma_I^+ + \delta_I \beta_I^+) \beta_I^+, \hspace{1cm} \alpha_I = \frac{1}{\sqrt{\alpha_I^2} |a_I|}, \hspace{1cm} u_B = \frac{1}{\sqrt{\alpha_I^2} \alpha_I} \beta_I, \hspace{1cm} v_B = \frac{1}{\sqrt{\alpha_I^2} \delta_I}. \] (1.33b)

In the case $\delta = 0$ we express (1.27b) by (1.29b):
\begin{align*}
    \alpha_I &= 0, \hspace{1cm} \beta_I = -v_B, \hspace{1cm} \gamma_I = u_B, \hspace{1cm} \delta_I = 0, \\
    \beta_I^+ &= v_B^2 \beta_I^+ u_B, \hspace{1cm} |a_I| = |a_B|. \hspace{1cm} (1.34a)
\end{align*}
The inverse of (1.34a) are obvious. We note only

$$b_B = -\beta_1 b_1 \gamma_1^+.$$  (1.34b)

Proof: Formulae (1.33a) are obtained by substituting in (1.27b) the inverse formulae of (1.28b). The remaining equations are verified in a similar fashion.

We mention two special cases. First the Iwasawa decomposition of $\tilde{n}_x$:

$$\tilde{n}_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = k_x n_x a(x),$$  (1.35)

where

$$k_x = \frac{1}{\sqrt{1+x^2}} \begin{bmatrix} 1 & x \\ -x^+ & 1 \end{bmatrix}, \quad |a(x)| = 1/(1+x^2);$$

then the Bruhat decomposition of $k$:

$$k = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = n_{x(k)} n_{b(k)} a(k) m(k), \quad (\alpha^2 = \delta^2 \neq 0)$$  (1.36a)

where

$$x(k) = \frac{1}{\alpha^2} \beta \delta^+, \quad b_+^+(k) = -\beta_2^+, \quad |a(k)| = 1/\alpha^2,$$

$$m(k) = \frac{1}{\sqrt{\alpha^2}} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$  (1.36b)

In the case $x = \delta = 0$ instead of (1.36a) we have

$$k = \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & -\beta \end{bmatrix} \in M', \quad (\beta^2 = \gamma^2 = 1).$$  (1.36b)

It is also interesting to note that almost every element of $K$ can be decomposed in the form

$$k = k_x m \quad (\alpha^2 = \delta^2 \neq 0).$$  (1.37)

Proposition 1.5. Let $k$ be as in (1.36a), then in (1.37)

$$x = \frac{1}{\alpha^2} \beta \delta^+, \quad m = \frac{1}{\sqrt{\alpha^2}} \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}. $$  (1.38)

Proof: It suffices to apply the Bruhat decomposition of $k_x$

$$k_x = n_{x(k)} b_x a_x,$$  (1.39a)

where

$$b(x) = -\frac{1}{1+x^2} x, \quad |a_x| = 1+x^2. $$  (1.39b)

Note. Elements $k_x$ may be considered as forming the factor space $K/M$.

1.E. The Haar measure. We shall summarize here some results of ref. [4] concerning the invariant measure on $G$ and its expression in terms of the invariant measures of its
subgroups in the Iwasawa decomposition. Since the group $G$ is unimodular, its Haar measure is both left and right invariant; the measures of the non-unimodular factors are chosen to be left invariant.

Let $dk$ be the Haar measure on $K$ normalized by

$$\int_{K} dk = 1.$$  \hspace{1cm} (1.40)

Set further

$$da \equiv \frac{d|a|}{|a|}, \quad dn_b = db_1 db_2 db_3 db_4 \equiv db.$$  \hspace{1cm} (1.41)

Then the Haar measure on $G$ has the form

$$dg = d(kn,a) = dk db da.$$  \hspace{1cm} (1.42)

We shall also need the expression for $dk$ in terms of the factors $k_x$ and $m$ of (1.37); it is

$$dk = d(k,m) = \frac{6}{\pi^2} \frac{dx}{(1 + x^2)^4} dm,$$  \hspace{1cm} (1.43)

where

$$dx \equiv dx_1 dx_2 dx_3 dx_4 = d\tilde{n}_x$$

and $dm$ is the normalized Haar measure on $M$:

$$\int_{M} dm = 1.$$

2. Elementary induced representations

2.A. The unitary irreducible representations of $M$. The unitary irreducible representations of $M (= \text{SU}(2) \times \text{SU}(2))$ are characterized by two positive integers, say $m_1$ and $m_2$. Following [1] we realize these representations in the space $V^l (l \equiv (m_1, m_2))$ of homogeneous polynomials of two complex two-vectors, i.e.,

$$V^l \equiv \{ \varphi: C^2 \times C^2 \to C \mid \varphi(\lambda w^+, \mu z) = \lambda^{m_1} \mu^{m_2} \varphi(w^+, Z), \lambda, \mu \in C \},$$  \hspace{1cm} (2.1)

where

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in C^2, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in C^2,$$

and we use the hermitian conjugate two-vector $w^+$ (rather than $w$) for convenience. Obviously $\dim V^l = (m_1 + 1)(m_2 + 1)$.

The UIR $D^l$ is given by the formula [1]:

$$[D^l(m)\varphi](w^+, z) \equiv \varphi(w^+ u, v^+ z), \quad m = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}.$$  \hspace{1cm} (2.2)

(From now on we shall not mark transposition of complex two-vectors, e.g. instead of $w^+ \equiv \overline{w}$ we shall write $\overline{w}$.)
We note for future reference
\[ m_1 m_2 \varphi(\bar{w}u, v^+ z) = \left( \frac{\partial}{\partial \bar{w}} \right)^{m_1} \left( \frac{\partial}{\partial \bar{v}^+} \right)^{m_2} \varphi(\bar{w}', z') \]
\[ = \varphi \left( \frac{\partial}{\partial \bar{w}'}, \frac{\partial}{\partial \bar{v}'^+} \right) (\bar{w}u \bar{w'})^{m_1} (\bar{v}'^+ z)^{m_2} = \left( \frac{\partial}{\partial \bar{v}'^+} \right)^{m_1} \left( \frac{\partial}{\partial \bar{w}'} \right)^{m_2} \varphi(\bar{w}', \bar{v}') \]
where \( m \equiv q_2, \frac{\partial}{\partial z} = \left( \frac{\partial \bar{v}'^+}{\partial z_1}, \frac{\partial \bar{w}'}{\partial z_2} \right) \). It follows from this equation that the sesquilinear form
\[ \langle \varphi, \varphi \rangle = \bar{\varphi} \left( \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial \bar{z}} \right) \varphi(\bar{w}, z) \frac{1}{m_1 m_2!} \]
defines a \( D' \)-invariant scalar product in \( V' \).

Every representation of \( M \) is a representation of \( SO(4) \) and we shall use sometimes the standard labelling of the representations of \( SO(4) \)
\[ [l_1, l_2] \equiv \left[ \frac{m_2 - m_1}{2}, \frac{m_2 + m_1}{2} \right]. \]
The numbers \( l_1, l_2 \) are simultaneously integer or half-integer. In the latter case the representations of \( SO(4) \) are double-valued (spinor) representations.

In order to find the infinitesimal generators of the representation \( D' \) we use (2.2') and obtain
\[ X_k \varphi(\bar{w}, z) = \frac{1}{2} \bar{w}q_k \frac{\partial}{\partial \bar{w}} \varphi(\bar{w}, z), \]
\[ Y_k \varphi(\bar{w}, z) = \frac{1}{2} z'q_k \frac{\partial}{\partial z} \varphi(\bar{w}, z). \]

The second order Casimir operator of the representation \( D' \) is
\[ \Omega(l) = -\frac{1}{2} (X_k^2 + Y_k^2) = -\frac{1}{2} [m_1(m_1 + 1) + m_2(m_2 + 1)]. \]

In our considerations very important role along with \( D' \) will play its so-called "mirror image" representation \( D'_{\bar{m}} \) acting in the same space \( V' \) by the formula
\[ [D'_{\bar{m}}(m) \varphi](\bar{w}, z) \equiv [D'(m^*) \varphi](\bar{w}, z) = \varphi(\bar{w}v, u^+ z) \]
where \( m^* \) is the Weyl conjugated matrix of \( m \):
\[ m^* \equiv \alpha m \alpha^{-1} = \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}, \text{ if } m = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}. \]

The representations \( D' \) and \( D'_{\bar{m}} \) are not equivalent except for \( m_1 = m_2 \). The mirror image \( D'_{\bar{m}} \) is equivalent to the representation \( \bar{D}' \), where \( \bar{l} = (m_2, m_1) \) for \( l = (m_1, m_2) \). This is seen by the following...
PROPPOSITION 2.1. The intertwining operator of $D^i$ and $D^i_d$ is given by

$$B: V^i \rightarrow V^i,$$  \hspace{1cm} (2.8a)

$$(B\varphi)(\overline{w}, z) \equiv \varphi(ze, \overline{e}w), \quad \varphi \in V^i;$$ \hspace{1cm} (2.8b)

its inverse

$$B^{-1}: V^i \rightarrow V^i$$

is given by the same formula (2.8b) for $\varphi \in V^i$.

Proof: The only thing left is to check using the definitions:

$$BD^i = D^i_d B \quad \text{or} \quad D^i_d B^{-1} = B^{-1}D^i_d.$$ \hspace{1cm} (2.8c)

Remark. We note that if $m \mapsto M \in SO(4)$ as in (1.32), then $m^* \mapsto I_x M I_x \in SO(4)$ ($I_x \in O(4)$, for $x = (x, x_4) \in \mathbb{R}^4$, $I_x x = (-x, x_4)$), i.e. the mirror image transformation of $SO(4)$ (cf. [21]). That is the reason that we called the Weyl conjugated representation the mirror image representation.

2.B. Definition of the induced representations. We shall give two equivalent realizations of the elementary induced representations (induced by the parabolic subgroup $MNA$) (cf. [14, 4]).

Let $V^i$ be the finite-dimensional Hilbert space (2.1) of the UIR $D^i$ of $M$. Let $c \in C$, $\chi \equiv [l, c] \equiv (m_1, m_2; c) \equiv [l_1, l_2; c]$, (2.9)

and $D^x$ is the finite-dimensional representation of $MA$ given by

$$D^x(ma) \equiv |a|^{-2-c} D^i(ma).$$ \hspace{1cm} (2.10)

Then we define

$$\mathcal{C}_x \equiv \{ f: G \times C \times C \rightarrow C \mid f \in C^\infty(G), \ f(g) \in V^i, f(gma, \overline{w}, z)$$

$$= [D^x(m^{-1}a^{-1})] f](g, \overline{w}, z) = |a|^{2+c} f(g, \overline{w}u, vz) \}.$$ \hspace{1cm} (2.11)

The representation $\mathcal{X}^x$ of $G$ induced by the finite-dimensional representation $D^x$ of $MNA$ (N being represented trivially) is defined by

$$[\mathcal{X}^x(g)f](g') \equiv f(g^{-1}g'), \quad g, g' \in G, \ f \in \mathcal{C}_x.$$ \hspace{1cm} (2.12)

Each $\mathcal{X}^x$ is called an elementary induced representation of $G$.

The $K$-invariant scalar product in $\mathcal{C}_x$ is defined by setting

$$(\mathcal{C}_x \times \mathcal{C}_x \equiv \frac{\pi^2}{3!} \int \frac{dK}{K} \langle \mathcal{X}_1(k), \mathcal{X}_2(k) \rangle,$$ \hspace{1cm} (2.13a)

(factor $\pi^2/3!$ is introduced for convenience). Using the factorization of $dK$ (1.43), covariance properties of $\mathcal{X}$ and $M$-covariance of $\langle, \rangle$ we obtain

$$\langle \mathcal{X}_1, \mathcal{X}_2 \rangle_{\mathcal{C}_x} = \int_{\mathbb{R}^4} \frac{dx}{(1+x^2)^2} \langle \mathcal{X}_1(kx), \mathcal{X}_2(kx) \rangle.$$ \hspace{1cm} (2.13b)
The representation $\mathcal{T}^x$ is continuous with respect to the topology defined by (2.13). Another realization of the elementary induced representations is the noncompact picture (there is also the so called compact picture). Let

$$C_x \equiv \left\{ f : \mathbb{R}^4 \times C^2 \times C^2 \to C \mid f \in C^\omega(\mathbb{R}^4), f(x) \in V^l \right\}$$

where $H_{kl}$ is a homogeneous polynomial of degree $k, m_1, m_2$ in the first, second and third variable, respectively:

$$\omega x = (x, -x_4)/x^2 \quad \text{for} \quad x = (x, x_4). \quad (2.14b)$$

The representation $T^x$ is defined by

$$[T^x(g)f](x, \bar{w}, z) = f(x - x', \bar{w}, z), \quad (2.16a)$$

(c) dilatations

$$[T^x(m)f](x, \bar{w}, z) = f(A^{-1}x, \bar{w}u, v^+z), \quad (2.16b)$$

(d) Weyl inversion (then for special conformal transformations we use $n_b = \omega n_{I_b}\omega$)

$$[T^x(\omega)f](x, \bar{w}, z) = \frac{1}{(x^2)^{2+c}} f \left( \omega x, \frac{-\bar{w}x}{\sqrt{x^2}}, \frac{-xz}{\sqrt{x^2}} \right). \quad (2.16d)$$

In (2.16d) we have used

$$\omega^{-1}\bar{n}_x = \bar{n}_{ax} n_{I,x} a(x, \omega)^{-1} m(x, \omega)^{-1} \quad (2.16e)$$

where $|a(x, \omega)| = x^2, m(x, \omega) \equiv -\frac{1}{\sqrt{x^2}} \begin{bmatrix} x & 0 \\ 0 & x^+ \end{bmatrix}$.

We note that the asymptotic behaviour (for $x \to \infty$) of the functions of $C_x$ guarantees the smoothness of the right-hand side of (2.16d) for $x \to 0$.

The operator which realizes the equivalence between $\mathcal{T}^x$ and $T^x$ is

$$A_0 : C_x \to C_x, \quad (A_0 f)(x, \bar{w}, z) = f(\bar{n}_x, \bar{w}, z), \quad (A_0^{-1} f)(g, \bar{w}, z) \equiv |a|^{2+c} f(x, \bar{w}u^+, vz), \quad g = \bar{n}_x n_{am}.$$
The G-invariant scalar product in $C_x$ is defined by setting (for $c = i\sigma, \sigma \in R$)

$$ (f_1, f_2)_{C_x} = \int_{R^4} \langle f_1(x), f_2(x) \rangle dx. \quad (2.18) $$

The representation $T^x$ is continuous with respect to the topology defined by (2.18). The operator $A_0$ is isometric. Indeed using (1.35) and (2.13) we obtain

$$ (A_0 f_1, A_0 f_2)_{C_x} = \int_{R^4} \langle (A_0^{-1} f_1)(x), (A_0^{-1} f_2)(x) \rangle dx \quad \frac{dx}{(1 + x^2)^4} = (A_0^{-1} f_1, A_0^{-1} f_2)_{C_x}. \quad (2.19) $$

The spaces $C_x$ and $C_{\bar{x}}$ are not complete as normed linear spaces. Their completion with respect to their scalar products will be denoted by $\mathcal{H}_x$ and $H_{\bar{x}}$, respectively. (Note however that they are complete with respect to some Fréchet space topologies—cf. [5a].)

Along with $T^x$ we shall use its conjugate representation $T_{\bar{x}}$ and its Weyl conjugate representation $T^{\bar{x}a}$. Here for $\chi = [l, c]$

$$ \tilde{\chi} \equiv [\bar{c}, -l], \quad \chi_a \equiv [l_a, -c], \quad (2.9') $$

where $l_a$ denotes that the inducing representation of $M$ is $D_{l_a}^*$. 

2.C. Infinitesimal generators and Casimir operators of the elementary representations. We shall write the infinitesimal generators in the noncompact picture for the subgroups $\tilde{N}, M, A, N$. Using (2.16) we have

(a) translations

$$ T_\mu f(x, \bar{w}, z) = \frac{\partial}{\partial x_\mu} \langle T^x(n_x) f(x, \bar{w}, z) \rangle_{x'=0} = \frac{\partial}{\partial x_\mu} f(x-x', \bar{w}, z)_{x'=0}, \quad T_\mu = -\nu_\mu \equiv -\frac{\partial}{\partial x_\mu}; \quad (2.20a) $$

(b) rotations: for $m = \left[ 1 + \frac{1}{2} \delta^i_k q_k \begin{array}{cc} 0 & 0 \\ 0 & 1 + \frac{1}{2} \delta^{i'}_{k'} q_{k'} \end{array} \right], \delta_k, \delta^{i'}_{k'} \ll 1$,

$$ X_k f(x, \bar{w}, z) = \frac{\partial}{\partial \delta_k} \langle T^x(m)f(x, \bar{w}, z) \rangle_{y'=0} = \frac{1}{2} \bar{w} q_k \frac{\partial}{\partial \bar{w}} + \frac{1}{2} x_k \nabla_4 - \frac{1}{2} x_4 \nabla_k - \varepsilon_{kij} x_i \nabla_j f(x, \bar{w}, z), \quad (2.20b') $$

where $\bar{\delta} = (\delta_1, \delta_2, \delta_3)$;

$$ Y_k = -\frac{1}{2} z' q_k \frac{\partial}{\partial z} - \frac{1}{2} x_k \nabla_4 + \frac{1}{2} x_4 \nabla_k - \varepsilon_{kij} x_i \nabla_j; \quad (2.20b'') $$

(c) dilatations:

$$ Df(x, \bar{w}, z) \equiv \frac{\partial}{\partial |x|} \langle T^x(a)f(x, \bar{w}, z) \rangle_{|a|=1} = (-2 - c - x \nabla)f(x, \bar{w}, z). \quad (2.20c) $$
(d) To find the infinitesimal generators of *special conformal* transformations we first calculate (see [2]) using (1.28) (cf. (1.32))

\[ n_b^{-1} \tilde{\eta}_\nu = \tilde{\eta}_\nu n_b^{-1} a'^{-1} m'^{-1}, \]  

\[ x' = \frac{x - x^2 b}{\sigma}, \quad |a'| = \sigma, \quad m' = \frac{1}{\sqrt{\sigma}} \begin{bmatrix} 1 - \frac{x b^+}{2} & 0 \\ 0 & 1 - \frac{x^+ b}{2} \end{bmatrix}, \]

\[ \sigma = 1 - 2bx + b^2 x^2 \quad (\sigma \neq 0 \text{ since } b \to 0). \]

Thus we have

\[ C_\mu(x, \bar{w}, z) = \frac{\partial}{\partial b^\mu} \left( T^n(n_b)f \right)(x, \bar{w}, z)_{b=0} \]

\[ = \frac{1}{m_1! m_2!} \frac{\partial}{\partial b_\mu} \left[ \frac{1}{\sigma^{2+}+1} \left( \bar{w}(1-x^2 b^+) \frac{\partial}{\partial w^+} \right) \left( \frac{\partial}{\partial z^+} (1-b^+ x) z \right) \right]_{b=0} = (C_\mu^0 + C_\mu^1) f(x, \bar{w}, z), \tag{2.20d} \]

where

\[ C_\mu^0 = 2(2+c) x_\mu + (2x_\mu x_\nu - \delta_{\mu\nu} x^2) \nabla_\nu, \]

\[ C_\mu^1 = 2l_2 x_\mu - x_\nu \bar{w} q_\nu q_\mu^* - q_\mu \frac{\partial}{\partial w} - x_\nu z q_\nu q_\mu \frac{\partial}{\partial z}. \tag{2.21} \]

Using homogeneity of the representation functions we obtain from (2.21):

\[ C_4 = -x_k \left( \bar{w} q_k \frac{\partial}{\partial w} + z q_k \frac{\partial}{\partial z} \right), \]

\[ C_j = x_k \left( \bar{w} q_j \frac{\partial}{\partial w} + z q_j \frac{\partial}{\partial z} \right) - \varepsilon_{jkli} x_k \left( \bar{w} q_l \frac{\partial}{\partial w} - z q_l \frac{\partial}{\partial z} \right). \tag{2.22} \]

In accord with the general fact of operator (or Schur) irreducibility of elementary representations the Casimir operators are multiples of the unit operator; in particular,

\[ \mathcal{C}_2(\chi) = D^2 + CT + 4D - 2(X_k^2 + Y_k^2) \tag{2.23} \]

is given by

\[ \mathcal{C}_2(\chi) = \frac{1}{2} \left[ m_1(m_1+2) + m_2(m_2+2) \right] + c^2 - 4. \tag{2.24} \]

3. Properties of the elementary induced representations

3.A. Irreducibility of the elementary induced representations. We shall review some general properties of the elementary induced representations.

As already mentioned every EIR of \( G \) on \( C_x \) is operator irreducible in the sense of Schur's lemma. Nevertheless, as we shall see, some of EIR are topologically reducible in the sense that there exist nontrivial closed invariant subspaces in some of \( C_x \).
It will be shown in Section 5 that the representations labelled by
\[ \chi_{lm} (l \equiv (2l+n+1, n-1; -l-v-1) \equiv [n-l-1, l; -l-v-1], \quad (3.1a) \]
contain finite-dimensional invariant subspaces.

For fixed \( l, v \) and \( n \) there are three more reducible representations obtained from \( T_{lm} \) (equiv. \( T_{lm} \)) by a chain of intertwining maps, which establish pairwise partial equivalence. They are labelled as follows:
\[ \chi^+_{lm} \equiv (2l+v+n+1, v-1; -l+n-1) \equiv [l+1-n, l; l+v+1] = \tilde{\chi}_{lm}, \quad (3.1b) \]
\[ \chi^-_{lm} \equiv (2l+v+n+1, v-n; -l-1) \equiv [n-1, l+v; -l-1], \quad (3.1c) \]
\[ \chi^0_{lm} \equiv (v+n-1, 2l+v-n+1; l+1) \equiv [l+1-n, l+v; l+1] = \tilde{\chi}_{lm}. \quad (3.1d) \]

Since the Casimir operators of \( G \) are multiples of the identity for each elementary representation, they must have the same value for the representations (3.1). Indeed the Casimir (2.24) has the value
\[ \mathcal{C}_2(\chi) = (l+v-n+1)^2 + 2(l^2 + 2l + vn) - 3 \quad (3.2) \]
for all the representations (3.1).

The same value of the Casimir operator have two additional representations (for fixed \( l, v, n \)):
\[ \chi'^+_{lm} \equiv (2l+v+1, v-1; -l+n-1) \equiv [l+1-n, l+v; -l+1+n], \quad (3.3a) \]
\[ \chi'^-_{lm} \equiv (v-1, 2l+v+1; l+n-1) \equiv [l+1, l+v; l+1-n] = \tilde{\chi}'_{lm}. \quad (3.3b) \]

These representations are partially equivalent to (3.1), but are irreducible (see Sec. 5).

All nonexceptional elementary representations different from (3.1) are topologically irreducible [12, 13].

The importance of the elementary representations comes from the fact that every IR of \( G \) (in a certain sense) is equivalent either to an elementary IR or to an irreducible component of an elementary reducible representation. For more details see [4], Sec. 3.A (see also [19, 31]).

We shall use some facts about characters of EIR as given in [4] (cf. also [18]).

For infinitely differentiable functions \( \varphi \) of compact support on \( G \) the operator
\[ \mathcal{X}(\varphi) \equiv \int g \mathcal{X}(g) dg \quad (3.4) \]
is known to be trace class. Its trace
\[ \theta_\varphi (\varphi) \equiv \text{Tr} \mathcal{X}(\varphi) \quad (3.5) \]
is called the character of the EIR.

We introduce an auxiliary function
\[ F_{\alpha}(ma) \equiv |a|^h \int_{K \times N} \varphi(kmnak^{-1}) dk dn \quad (3.6) \]
with covariance condition
\[ F_\varphi(m'm a m'^{-1}) = F_\varphi(m a), \quad m' \in M'. \tag{3.7} \]

The character \( \theta_x \) is expressed in terms of \( F_\varphi \) according to
\[ \theta_x(\varphi) = \int_{M_\varphi} |a|^{-c} [\text{tr} D^1(m)] F_\varphi(m a) \, dm a. \tag{3.8} \]

The characters of conjugate or Weyl conjugate representations coincide:
\[ \theta_{x_0}(\varphi) = \theta_x(\varphi), \tag{3.9a} \]
\[ \theta_{x_0}(\varphi) = \theta_{x}(\varphi). \tag{3.9b} \]

Indeed, (3.10a) is proved by change of variables and use of (3.7) (cf. [4]), while for (3.10b) we need only to show that
\[ \text{tr} D^1_\omega = \text{tr} D^\omega \tag{3.10} \]
which is easily done in appropriate basis of \( V^l \) and \( V^\omega \).

3.B. Invariant sesquilinear forms on pairs of EIR. Suppose we have an invariant sesquilinear form \( L \) on the spaces \( C_{x_1} \times C_{x_2} \), given by \( (f_1 \in C_{x_1}, \, i = 1, 2) \)
\[ L(f_1, f_2) = \int_{\pi} \langle f_1(x), f_2(x) \rangle. \tag{3.11} \]

In writing (3.11) we have a restriction on \( x_1 \) and \( x_2 \), namely if \( x_1 = [l_1, c_1], \, x_2 = [l_2, c_2] \)
then either \( l_2 = l_1 \) or \( l_2 = l_1 \).

Checking invariance of (3.11), i.e. establishing
\[ L(T^{x_1}(g)f_1, T^{x_2}(g)f_2) = L(f_1, f_2) \tag{3.12} \]
for \( g = \omega \) we obtain
\[ \bar{c}_1 + c_2 = 0, \tag{3.13a} \]
\[ l_2 = l_1. \tag{3.13b} \]
(Invariance for \( g = \tilde{n} \in \tilde{N} \) is trivial and follows from (3.13) for \( g = a \in A \) and \( g = m \in M \).)

Denoting for \( \chi = [l, c] \)
\[ *\chi = [l, -\bar{c}] \tag{3.14} \]
we conclude:

**Proposition 3.1.** There exist an invariant sesquilinear form \( L \) on the product \( C_{*x} \times C_x \) given by (3.11) \( (f_1 \in C_{*x}, \, f_2 \in C_x) \).

**Remark.** In the case of unitary characters of \( A \) \( (c = i\sigma, \, \sigma \in \mathbb{R}) \) we have \( *\chi = [l, -\bar{c}] = [l, c] = \chi \) and so (3.15) defines an invariant scalar product in \( C_x \) since (cf. (2.18))
\[ L(f_1, f_2) = (f_1, f_2)_{C_x}. \]
(For nonunitary characters of \( A \) (2.13) is only \( K \)-invariant.) This scalar product gives rise to the so-called principal series of unitary irreducible representations—cf. [2, 14, 4].
4. Intertwining operators

4.A. Definition and construction of the intertwining operators. By a well known theorem (see [19], Vol. I, Sec. 4.5) a character determines every unitary irreducible representation uniquely (up to equivalence). Since, according to (3.10) the unitary principal series representations \( \chi = [l, c], \chi_{wa} \) and \( \chi \) have the same character, they must be unitarily equivalent. Therefore there should be operators intertwining these representations.

First we introduce the equivalence maps between the representations \( \mathcal{X}^{\chi} \) and \( \mathcal{X}^{\chi_{wa}} \) and between \( \mathcal{X}^{\chi_{x}} \) and \( \mathcal{X}^{\chi_{x_{wa}}} \):

\[
B_x: \mathcal{C}_{\chi_{wa}} \rightarrow \mathcal{C}_{\chi}, \tag{4.1a}
\]

\[
(B_x f)(g, \bar{w}, z) = (B \bar{f}(g))(\bar{w}, z) = f(g, ze, \bar{w}) \tag{4.1b}
\]

(cf. (2.8b)),

\[
B_{\chi_{wa}}: \mathcal{C}_{\chi_{wa}} \rightarrow \mathcal{C}_{\chi_{wa}} \tag{4.1c}
\]

and the action of \( B_{\chi_{wa}} \) is given by (4.1b) for \( f \in \mathcal{C}_{\chi} \).

Note that

\[
B_x B_{\chi_{wa}} = \text{id}_{\mathcal{X}^{\chi}}, \quad B_{\chi_{wa}} B_x = \text{id}_{\mathcal{X}^{\chi_{wa}}} \tag{4.1d}
\]

(4.1e)

where by id\( \chi \) is denoted the unit operator in the space \( \mathcal{X} \).

The intertwining properties of \( B_x \) and \( B_{\chi_{wa}} \)

\[
T^x B_x = B_x T^\chi, \quad T^\chi B_{\chi_{wa}} = B_{\chi_{wa}} T^\chi \tag{4.1e}
\]

are proved easily since these operators are reduced to the operator \( B \) and (4.1e) is reduced to (2.8c).

The intertwining operator

\[
A_x: \mathcal{C}_{\chi_{wa}} \rightarrow \mathcal{C}_{\chi} \tag{4.2a}
\]

which, exhibits the equivalence between \( \mathcal{X}^{\chi} \) and \( \mathcal{X}^{\chi_{wa}} \) [14] is defined by

\[
(A_x f)(g, \bar{w}, z) = \gamma(\chi) \int f(g o \bar{n}_x, \bar{w}, z) dx, \tag{4.2b}
\]

\[
\mathcal{X}^x A_x = A_x \mathcal{X}^{\chi_{wa}}. \tag{4.2c}
\]

(The constant \( \gamma(\chi) \) shall be fixed later.) This operator in the noncompact picture (counting the equivalence of the general and noncompact pictures we use the same letters):

\[
A_x: C_{\chi_{wa}} \rightarrow C_{\chi}, \quad T^x A_x = A_x T^{\chi_{wa}}, \tag{4.3a}
\]

can be represented as follows (cf. [4])

\[
(A_x f)(x_1, \bar{w}_1, z_1) \equiv (A_x f)(\bar{n}_x, \bar{w}_1, z_1)
\]

\[
= \gamma(\chi) \int D_o \left( \frac{1}{\sqrt{x_{12}}} \left[ \begin{array}{cc} x_{12} & 0 \\ 0 & x_{12} \end{array} \right] \right) f(x_2, \bar{w}_1, z_1) \frac{dx_2}{(x_{12})^2 + c} \tag{4.3b}
\]

\[
= \frac{\gamma(\chi)}{m_1! m_2!} \int \left( \frac{\partial}{\partial \bar{w}_1} \frac{\partial}{\partial \bar{w}_2} \right)^{m_1} \left( \frac{\partial}{\partial z_2} \frac{\partial}{\partial z_2} \right)^{m_2} f(x_1, \bar{w}_1, z_2) \frac{dx_2}{(x_{12})^2 + c + i \epsilon},
\]

\[
(4.3b)
\]
In deriving (4.3b) we have used (2.16) and changed variables \( x = \omega(x_2 - x_1) \).

Analogously is defined the inverse operator

\[
A_{\omega}: C_x(C_\omega) \to (C_\omega)_{x_0},
\]

and

\[
(A_{\omega}f)(\omega, \bar{w}, z) = \gamma(\omega) \int f(\omega \bar{w}, \bar{w}, z) \, dx,
\]

\( X_{x_0}A_{\omega} = A_{x_0}X_{x_0} \).

Analogously is defined the inverse operator

\[
A_{\omega}: (C_\omega)_{x_0} \to C_\omega,
\]

and

\[
(A_{\omega}f)(x_1, \bar{w}_1, z_1) = \frac{\gamma(\omega)}{m_1!m_2!} \int \left( \frac{\partial}{\partial w_2} \right)^{m_1} \left( \frac{\partial}{\partial z_2} \right)^{m_2} f(x_1, \bar{w}_1, z_1) \, dx_2.
\]

\( T^*A_{\omega} = A_{x_0}T^* \).

The constants \( \gamma(\omega) \) shall be fixed (not uniquely) by the condition that \( A_{\omega} \) is the inverse of \( A_x \). We obtain

\[
(A_xA_{\omega}f)(x_1, \bar{w}, z) = \gamma(\omega) \gamma(\omega) \int f(x_3, \bar{w}_1 x_3 x_3, \bar{w}_1 x_3 x_3, \bar{w}_1 x_3 x_3 z),
\]

\[
dx_2dx_3 = \frac{(-1)^{m_1+m_2}}{(l_1^2-c^2)![(l_2+1)^2-c]^2} \gamma(\omega) f(x_1, \bar{w}, z).
\]

(Note that for the principal series of unitary representations \( c = i\sigma, \sigma \in R \), so \( c^2 = -\sigma^2 < 0 \).) The necessary calculations are given in Appendix A. Setting

\[
\gamma(\omega) = \frac{(-1)^{(m_1+m_2)/2}c^2}{\pi^2} n(\chi)
\]

and requiring that the RHS of (4.5) equals \( f(x_1, \bar{w}, z) \), we obtain

\[
n(\omega)n(\omega) = \frac{(l_1^2-c^2)![(l_2+1)^2-c^2]},
\]

\( A_xA_{\omega} = id_c x \).

Next we define the intertwining operator of the representations \( T^0 \) and \( \tilde{T}^0 \):

\[
G_{\omega}: C_{\omega} \to C_x,
\]

\( G_{x} \equiv A_xB_{x_0}, \)

\( (G_{x}f)(x_1, \bar{w}_1, z_1) = \frac{\gamma(\omega)}{m_1!m_2!} \int \left( \frac{\partial}{\partial w_2} \right)^{m_1} \left( \frac{\partial}{\partial z_2} \right)^{m_2} f(x_2, \bar{w}_2, z_2) \, dx_2. \)

Further we note that the operator \( B_xA_{\omega}: C_{\omega} \to C_x \) is given by (4.7c) with \( \gamma(\omega) \) replaced by \( \gamma(\omega) \). Thus \( A_xB_\omega \) and \( B_xA_{\omega} \) are proportional and we shall set

\[
\gamma(\omega) = \gamma(\chi), \quad n(\omega) = n(\chi).
\]

In this way

\[
G_{\omega} \equiv A_xB_\omega = B_xA_{\omega},
\]

\( (G_{x}f)(x_1, \bar{w}_1, z_1) = \frac{1}{m_1!m_2!} \int G_x(x_2; \bar{w}_1, z_1; \bar{w}_2, z_2) f(x_2, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial z_2}) \, dx_2. \)
with
\[ G_x(x; \overline{w}_1, z_1; \overline{w}_2, z_2) = \frac{(-1)^{(m_1+m_2)/2}}{(2\pi)^2} n(\chi) \left( \frac{2}{x^2} \right)^{2+c} \left( \frac{\overline{w}_1}{\sqrt{x^2}} z_1 \right)^{m_1} \left( \frac{\overline{w}_2}{\sqrt{x^2}} z_2 \right)^{m_2}. \]  
(4.7d)

Analogously we define the intertwining operator of the representations \( T^\chi \) and \( T^{\tilde{\chi}} \)
\[ G_{x^\chi} : C_{x^\chi} \to C_{x^\chi}, \]
(4.9a)
\[ G_{x^\chi} = A_{x^\chi} B_x = B_{x^\chi} A_{\tilde{\chi}}, \]
(4.9b)
and the action of \( G_{x^\chi} \) is given by (4.7c) for \( f \in C_{x^\chi} \) and \( G_x \) replaced by \( G_{x^\chi} \); also \( G_{x^\chi}(x) \)
is given by (4.7d) with the substitutions
\[ n(\chi) \to n(\tilde{\chi}), \quad c \to -c, \quad x \to x^+. \]
The intertwining properties of \( G_x \) and \( G_{x^\chi} \)
\[ T^xG_x = G_x T^x, \quad T^{x^\chi}G_{x^\chi} = G_{x^\chi} T^{x^\chi} \]
(4.10)
follow easily from (4.1e), (4.3a), (4.4c).

It is also trivial to check that \( G_x \) is the inverse of \( G_{x^\chi} \):
\[ G_x G_{x^\chi} = A_x B_{x^\chi} B_x A_{x^\chi} = A_x A_{x^\chi} = id_C \]
(4.11)
using (4.1d) and (4.6c). Analogously,
\[ G_{x^\chi} G_{x^\chi} = id_{C_{x^\chi}}. \]
(4.11b)

The operators \( A_x, A_{x^\chi}, G_x \) and \( G_{x^\chi} \)
were considered till now for unitary characters of \( A \), i.e. for \( c \) imaginary. Using (1.35) we write
\[ \frac{1}{\gamma(\chi)} (A_x f)(g) = \int dx f(g\omega \tilde{n}_x) = \int f(g\omega k_x n_x a(x)) dx \]
\[ = \int f(g\omega k_x) \frac{dx}{(1+x^2)^{2-c}}. \]
(4.2d)

Since \( k_x \) varies in the compact manifold \( K \) we see that \( \frac{1}{\gamma(\chi)} A_x \) is well defined and analytic in \( c \) for \( \text{Re} c < 0 \). The same applies to \( \frac{1}{\gamma(\chi)} G_x \), while \( \frac{1}{\gamma(\chi)} A_{x^\chi} \) and \( \frac{1}{\gamma(\tilde{\chi})} G_{x^\chi} \)
are well defined and analytic for \( \text{Re} c > 0 \). All those operators can be extended to meromorphic functions in the entire complex plane \( c \) (with poles on the real axis). This analytic continuation will be carried out in the next subsection.

4. B. Harmonic analysis of the intertwining operator \( G_x \). The harmonic analysis of \( G_x \)
on the parabolic subgroup \( \tilde{N}AM \) is reduced to the harmonic analysis on \( \tilde{N} \) and \( M \) since \( G_x(x) \) is irreducible with respect to \( A \) being homogeneous in \( x \).
First one performs the Fourier transform of $G_{\chi}(x)$ on $N \cong \mathbb{R}^{2\lambda}$ defining

$$
\tilde{G}_{\chi}(p) \equiv \int G_{\chi}(x)e^{-ipx} dx.
$$

We use the integration formula

$$
\int \left( \frac{2}{x^2} \right)^{2+\alpha} e^{-ipx} dx = (2\pi)^2 \frac{I(-\alpha)}{I(2+\alpha)} \left( \frac{p^2}{2} \right)^{1+\alpha}
$$

valid for $-1 < \text{Re} \alpha < 0$, extending it by analytic continuation to all non-integer $\alpha$. We introduce the notation

$$
\zeta_{1\mu} \equiv \frac{1}{\sqrt{2}} \tilde{w}_1 q_{\mu} \tilde{z}_2, \quad \zeta_{2\mu} \equiv \frac{1}{\sqrt{2}} \tilde{w}_2 q_{\mu} \tilde{z}_1.
$$

We obtain for (4.12)

$$
\tilde{G}_{\chi}(p; \tilde{w}_1, z_1; \tilde{w}_2, z_2) = \frac{(-1)^{2l_1 n(\chi)}}{(2\pi)^2} (\zeta_{1\nu} \nabla_{\nu})^{m_1}(\zeta_{2\nu} \nabla_{\nu})^{m_2} \int e^{-ipx} \left( \frac{2}{x^2} \right)^{2+c+l_2} dx
$$

$$
= \frac{(-1)^{2l_1 n(\chi)}}{(2\pi)^2} \frac{I(-c-l_2)I(c+l_2+1)}{I(2+c+l_2)} \sum_k \frac{(l_2-l_1)!((l_2+l_1)!(\zeta_1 \tilde{z}_2)^{l_2-l_1-k}}{(l_2-l_1-k)!(2l_1+k)!k!I(c-l_1-k+1)}
$$

$$
\cdot (\zeta_1 p)^k (\zeta_2 p)^{2l_1+k} \left( \frac{p^2}{2} \right)^{c-l_1-k}
$$

$$
= \frac{(-1)^{l_1-l_1} n(\chi)(l_2-l_1)!I(c-l_1-c)}{I(2+c+l_2)} \left( \frac{1}{\sqrt{2}} p \tilde{z}^* \right)^{2l_1+l_1} \left( 2(\tilde{p}_{31})(p_{32}) \right)^{l_2-l_1}.
$$

$$
\cdot \left( \frac{p^2}{2} \right)^{c} P_{l_2-l_1}^{(c-l_1)}(\chi)
$$

where

$$
\tilde{3}_{1\mu} \equiv \frac{1}{\sqrt{2}} \tilde{w}_1 q_{\mu} \tilde{z}_1, \quad \tilde{3}_{2\mu} \equiv \frac{1}{\sqrt{2}} \tilde{w}_2 q_{\mu} \tilde{z}_1,
$$

(i = 1, 2),

$$
\zeta^* = \begin{cases} 
\zeta_1, & l_1 \leq 0, \\
\zeta_2, & l_1 \geq 0,
\end{cases}
$$

$$
\zeta^* \equiv 1 - \frac{p^2(\tilde{3}_{1\mu} \tilde{3}_{2\mu})}{(p_{31})(p_{32})}.
$$

Note that in (4.15a) the range of summation in $k$ depends on the sign of $l_1$ and is determined by the factor $1/(l_2-l_1-k)!(2l_1+k)!k!$.

In deriving (4.15) we have used the following relation between the $\zeta$ and the $3$ variables [17]:

$$
2(p\zeta_1)(p\zeta_2) = 2(\tilde{p}_{31})(p_{32}) - p^2(\tilde{3}_{1\mu} \tilde{3}_{2\mu}).
$$

We have also used an expression for the Jacobi polynomials (8.962.2 of ref. [9]):

$$
P_n^{(a,b)}(x) = \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{I(-\alpha-k)I(\alpha+\beta+n+k+1)}{I(-\alpha-n)I(\alpha+\beta+n+1)} \left( \frac{1-x}{2} \right)^k.
$$
To carry out the harmonic analysis of $\tilde{G}_x(p)$ on $M$ for fixed $p$, we must expand $\tilde{G}_x(p)$ in the eigenfunctions of the Casimir operator of the stability group $U$ of $p$. In order to perform this step we shall use the frame in $R^4$ in which

$$p = \left((0, p_4 \equiv |p| \equiv \sqrt{p^2})\right).$$

(4.20)

In this frame the group $U \cong SU(2)$ is generated by (see (1.16) and (2.5))

$$X_{jk} = \epsilon_{jki}(X_1 + Y_1) = \frac{1}{2} \epsilon_{jki} \left( \overline{w} \frac{\partial}{\partial w} - zq_i \frac{\partial}{\partial z} \right)$$

(4.21)

(rotations in the $(j, k)$-plane leaving $p$ invariant). Then the Casimir of the stability group is the squared "spin operator":

$$S^2 = -\frac{1}{2} X_{jk} X^{jk} = -\frac{1}{2} \left( \frac{\partial}{\partial z} \right)^2 \left( \frac{\partial}{\partial w} \right)^2 - \frac{1}{2} \left( \frac{\partial}{\partial w} \right)^2 \left( \frac{\partial}{\partial z} \right)^2 + \frac{1}{4} \left( \frac{\partial}{\partial w} - z \frac{\partial}{\partial z} \right)^2.$$  

(4.22)

We denote for convenience the eigenvalues of $S^2$ by $s(s+1)$. For the eigenfunctions of $S^2$ we obtain (the calculations are done as in [17] since the Lie algebra of the stability subgroups of $M$ and $SO^+(3,1)$ may be identified)

$$F^{ls}(p; \overline{w}_1, z_1; \overline{w}_2, z_2) = (\hat{p} \hat{\zeta}^*)^{2l_1} \Pi^{ls}(p; \overline{w}_1, z_1; \overline{w}_2, z_2),$$

(4.23a)

$$\Pi^{ls}(p; \overline{w}_1, z_1; \overline{w}_2, z_2) = (-1)^{s-l_1} A_{ls} (2(\hat{p} \hat{\zeta}_1)(\hat{p} \hat{\zeta}_2))^{l_1-l_{s-1}} P^{(2l_1, l_{s-1})}(x)$$

(4.23b)

$s = |l_1|, |l_1|+1, \ldots, l_2$.

The operators $F^{ls}$ has the following orthogonality property

$$F^{ls} \circ F^{l's} : V^l \rightarrow V^{l'},$$

(4.24a)

Indeed

$$F^{ls} \circ F^{l's} = \frac{1}{s'(s'+1)} F^{ls} \circ F^{l's} = \frac{1}{s'(s'+1)} (S^2 F^{ls}) \circ F^{l's} = \frac{s(s+1)}{s'(s'+1)} F^{ls} \circ F^{l's} = 0, \quad s \neq s'.$$

The crucial point in deriving the last formula is that if $l = [l_1, l_2]$, then $\tilde{l} = [-l_1, l_2]$ and $F^{ls}$ depends on $|l_1|$ (not on $l_1$).

Let us denote

$$\pi^{ls} : V^l \rightarrow V^{l'}, \quad \pi^{ls} \equiv F^{ls} \circ F^{l's}.$$  

(4.24b)

The $\pi^{ls}$ are orthogonal:

$$\pi^{ls} \circ \pi^{l's} = F^{ls} \circ F^{l's} \circ F^{l's} \circ F^{ls} = 0, \quad s \neq s'.$
We shall fix the constant $A_{ls}$ in (4.23) by requiring
\[ \sum_{s=|l_1|}^{l_2} \pi^{ls} = \text{id}_V. \]  
(4.25)

For this we first decompose $\tilde{G}_x$ in $F^{ls}$ using the formula
\[
\tilde{G}_x(p) = \frac{\Gamma(\beta + l + 1)}{\Gamma(\alpha + \beta + l + 1)} \sum_{s=|l_1|}^{l_2} \frac{(2s + \beta + 1) \Gamma(\alpha + \beta + l + s + 1) \Gamma(\alpha + l - s)}{(l - s)! \Gamma(\beta + l + s + 2)} \cdot p_{ls}^{(0,\beta)}(x). 
\]
(4.26)

We obtain using (4.15b) and (4.26)
\[
\tilde{G}_x(p) = \frac{(l_2 + |l_1|)! (l_2 - |l_1|)! n(x) \Gamma(1 - c + l_2)}{(|l_1| - c) \Gamma(2 + c + l_2)} \left( \frac{p^2}{2} \right)^c. 
\]
(4.26)

Then we use
\[
\tilde{G}_x(p) \tilde{G}_z(p) = \text{id}_V \cdot (l_2 + |l_1|)! (l_2 - |l_1|)!^2 \sum_{s=|l_1|}^{l_2} \frac{(2s + 1)^2 F^{ls}(p) F^{lz}(p)}{(l_2 + s + 1)! (l_2 - s)! A_{ls}}.
\]

where we have accounted for (4.6b), (4.8), (4.24) and
\[
\frac{\Gamma(\alpha + k)}{\Gamma(\alpha + n)} = (-1)^{k+n} \frac{\Gamma(1 - \alpha - n)}{\Gamma(1 - \alpha - k)}.
\]

We obtain (4.25) by setting
\[
A_{ls} = \frac{(l_2 + |l_1|)! (l_2 - |l_1|)! (2s + 1)}{(l_2 + s + 1)! (l_2 - s)!} = A_{ls}
\]
(4.27)

(we fix the phase factor consistently with the case of symmetric tensor representations, cf. [4]).

Finally we obtain
\[
\tilde{G}_x(p) = \frac{n(x) \Gamma(1 - c + l_2)}{(|l_1| - c) \Gamma(2 + c + l_2)} \left( \frac{p^2}{2} \right)^c \sum_{s=|l_1|}^{l_2} \alpha_{st}(c) F^{ls}(p),
\]
(4.28a)

where
\[
\alpha_{st}(c) = (-1)^{s-|l_1|} \frac{\Gamma(c + s + 1) \Gamma(c - s)}{\Gamma(c + |l_1| + 1) \Gamma(c - |l_1|)}.
\]
(4.28b)

Note that in the symmetric tensor case ($l_1 = 0$) $F^{ls}$ reduces to $\Pi^{ls}$ which are orthogonal projections if we identify the spaces $V^l$ and $\bar{V}^l$ ($l = \bar{l}$); also $\pi^{ls}$ reduces to $\Pi^{ls}$ since $\pi^{ls} = F^{ls} \bar{F}^{ls} = \Pi^{ls} \Pi^{ls} = \Pi^{ls}$ in this case.
The normalization condition (4.6b) (using (4.8))

\[ n(x)n(\chi) = (l_1^2 - c^2)[(l_2 + 1)^2 - c^2] \]  

(4.29)
does not fix \( n(x) \) uniquely and there seems to be no universal choice of \( n(x) \) equally suited for all purposes. Following [14] we can choose \( n(x) \) to be a meromorphic function of \( c \) with all zeroes in the right half-plane and poles in the left-hand plane. The simplest choice of this type is given by

\[ n_+ (x) \equiv (1 + c + l_2) \frac{I(1 + c + |l_i|)}{I(|l_i| - c)}. \]  

(4.30)

Another choice of normalization

\[ n_0 (x) = \frac{(|l_i| - c)I(2 + c + l_2)}{I(1 - c + l_2)} \]  

(4.31)
is particularly convenient in writing some differential identities between sesquilinear forms for exceptional representations (see Section 5.E below).

A third choice is appropriate in the study of exceptional points \( \chi_{\text{inv}} \) (cf. (3.1)):

\[ n_- (x) = (|l_i| - c)(l_2 + 1 - c). \]  

(4.32)

We shall use the notation \( G^+ \), \( G^0 \) and \( G^- \) for the intertwining operators \( G \) with normalization (4.30), (4.31) and (4.32), respectively.

It is easily seen in the \( p \)-space picture that the operator \( G^+ \) is a meromorphic function of \( c \) and is not defined for \( \chi_{\text{inv}} \) (3.1a) and \( \chi_{\text{inv}}' \) (3.1c). Analogously the operator \( G^0 \) is not defined for \( \chi = \chi_{\text{inv}} \) and \( \chi_{\text{inv}}' \) (3.1d) and \( G^- \) is not defined for \( \chi = \chi_{\text{inv}} \) (3.1b) and \( \chi_{\text{inv}}' \). (For \( G^- \) it is appropriate to use the \( x \)-space picture.)

Thus the intertwining operator \( G_{\chi} \) is defined for every \( \chi \) (using the appropriate normalization) and we can conclude that the representations \( T^x \) and \( T^{\tilde{x}} \) are partially equivalent. In fact unless \( \chi \) is some exceptional representation of the type (3.1), \( T^x \) and \( T^{\tilde{x}} \) are equivalent and for unitary irreducible \( T^x \) they are unitarily equivalent for normalizations obeying (4.29).

4.C. Invariant sesquilinear forms involving \( G_{\chi} \). We saw in Section 3.B that the scalar product (2.13) is \( G \)-invariant for imaginary \( c \). It is easy to check that (2.13) is not \( G \)-invariant for \( c \) other than imaginary. However, there may be other unitary representations given by a hermitian form on \( C_x \times C_{\tilde{x}} \):

\[ S_x(f_1, f_2) = \int \langle f_1(x), S_x'(x, y)f_2(y) \rangle \, dx \, dy. \]  

(4.33)

The invariance condition

\[ S_{\chi}(T^x(g)f_1, T^x(g)f_2) = S_{\chi}(f_1, f_2) \]  

(4.34)

for \( g = n_x, g = m, g = a, \) and \( g = \omega \) gives

\[ S_{\chi}(x, y) = S_{\chi}(x - y), \]  

(4.35a)

\[ S_{\chi}(Ax)D^t(m) = D^t(m)S_{\chi}(x), \]  

(4.35b)
\[ S_x(|a|) = |a|^2 \text{He}^{-4} S_x(x), \]  
\[ \frac{1}{(x^2)^{2-\epsilon}} \cdot \frac{1}{(y^2)^{2-\epsilon}} D^i(m_x^+) S_x(x-y) D^i(m_y) = S_x(x-y), \]

where
\[ m_z = \frac{1}{\sqrt{z^2}} \begin{bmatrix} z^+ & 0 \\ 0 & z^- \end{bmatrix}. \]

If we consider (4.35d) for \( x = (0, x_4 > 0), y = (0, y_4 > 0) \) we obtain using (4.35c)
\[ \left( \frac{x_4}{y_4} \right)^{\epsilon - \epsilon} S_x(x-y) = S_x(x-y) \quad \text{or} \quad \bar{c} = c. \]

Further we notice that properties (4.35) (with \( c = \bar{c} \)) are those of an intertwining operator of the representations \( T^x \) and \( T^x_1 \) with \( x = [I, c] \) and \( x_1 = [I, -c] \). But we know that \( T^x \) is equivalent to \( T^{x_0} \) and \( T^{x_1} \). Thus we see that the hermitian form \( S \) exists iff \( \bar{c} = c \).

\[ \text{(4.36)} \]

Then \( \bar{x} - [\bar{l}, -c] = [I, -c] = x_1 \) and identifying \( V^l \) and \( V^{\bar{l}} \) (when \( \bar{l} = l \)) we obtain:
\[ S_x(x) = G_{\bar{x}}(x). \]

We shall also use the standard approach to the problem [14]. One starts with the identity
\[ \frac{1}{(x^2 y^2)^{2-\epsilon}} D^{i_1}(m_x^+) A^l_{\omega x}(\omega x - \omega y) D^i(m_y) = A^l_{\omega x}(x-y) \]  
\[ \text{(4.39)} \]

using
\[ D^{i_1}(m) = B_{\omega x} D^{\bar{l}}(m) B_{\bar{x}} \]
\[ \text{(4.40)} \]

to obtain (multiplying (4.39) on the left with \( B_{\bar{x}}^{\dagger} \))
\[ \frac{1}{(x^2 y^2)^{2-\epsilon}} \cdot \frac{1}{(y^2)^{2-\epsilon}} D^i(m_x^+) G_{\bar{x}}(\omega x - \omega y) D^i(m_y) = G_{\bar{x}}(x-y) \]
\[ \text{(4.41)} \]

and so \( S_x \) exists iff \( \bar{l} = l \) and \( S_{\bar{x}} = G_{\bar{x}} \).

Identifying \( V^{\bar{l}} \) with \( V^l \) (for \( \bar{l} = l \)), so that \( B_{\omega x} \) and \( B_{\bar{x}} \) are operators in \( V^l \) which will be denoted in compliance with (2.8) by \( B \) and \( B^{-1} \) respectively, we may rewrite (4.40) as
\[ D^{l_1}(m) = B D^l(m) B^{-1}. \]
\[ \text{(4.42)} \]

On the other hand
\[ D^{l_1}(m) = D^l(\omega m \omega^{-1}) = D^l(\omega) D^l(m) D^l(\omega)^{-1} \]
\[ \text{(4.43)} \]

where we have introduced \( D^l(\omega) \), because in the case \( \bar{l} = l \) it can be defined [14]. Comparing (4.42) and (4.43) we see that up to some unitary transformation commuting with \( D^l \) we must define
\[ D^l(\omega) = B, \]
\[ \text{(4.44a)} \]
\[ (D^l(\omega) \varphi)(\bar{w}, z) = \varphi(ze, \bar{w}). \]
\[ \text{(4.44b)} \]
Returning to the question of unitary representations besides the principal series we reach the following (well known) conclusion that necessary conditions for the existence of such representations are (for $\chi = [l, c]$)

$$\tilde{l} = l, \quad \tilde{c} = c.$$  \hspace{1cm} (4.45)

The necessary and sufficient conditions are found in [4] and the complementary series of unitary representations is singled out.

We note a useful relation between the forms (4.33) and (3.11). Observing that for $\tilde{l} = l, \tilde{c} = c$,

$$*\chi \equiv [l, -\tilde{c}] = [\tilde{l}, -c] \equiv \tilde{\chi}$$

and thus $f \in C_\chi = C*\tilde{\chi}$, $G_\tilde{\chi} = G*_\chi$ and $G*_\chi f \in C*_\chi$ we obtain

$$S_\chi(f_1, f_2) = L(f_1, G*_\chi f_2) \quad (\tilde{l} = l).$$  \hspace{1cm} (4.46)

Eq. (4.46) enables us to define an invariant sesquilinear form on $C_\chi \times C*_\chi$ for all $\chi$. Actually we shall define a form on $C*_\chi \times C_\chi$ ($f_1 \in C*_\chi, f_2 \in C_\chi$)

$$L_\chi(f_1, f_2) \equiv L(f_1, G*_\chi f_2) = L(G*_\chi f_1, f_2).$$  \hspace{1cm} (4.47)

For exceptional $\chi$ (cf. (3.1)) the form (4.47) is degenerate. However it shall be used in Section 5.C to define invariant sesquilinear forms on the invariant subspaces. Then (again in the case $\tilde{l} = l$) additional unitary representations will emerge.

5. Properties of exceptional elementary induced representations

5.A. Subrepresentations of reducible exceptional representations. The present section will be devoted to the study of the six families of exceptional representations, $\chi_{\text{fin}}^*, \chi_{\text{fin}}^{+\pm}$ and $\chi_{\text{fin}}^{\mp\mp}$ (see (3.1) and (3.3)). We shall use the shorthand notation $C_{\text{fin}}^*, C_{\text{fin}}^{+\pm}, C_{\text{fin}}^{\mp\mp}$ for the corresponding (noncompact picture) representation spaces, and $*C_{\text{fin}}^{+\pm\mp\pm}$ for the representation space of the respective $*\chi$ (3.14). Note that $*C_{\text{fin}}^{+\pm\mp\pm} = C_{\text{rep}}^{+\pm\mp\pm}$ ($p = 2l + 2 - n$), $*C_{\text{fin}}^{+\pm\mp\pm} = C_{\text{rep}}^{+\pm\mp\mp}$.

As was pointed out in Section 3.A the space $C_{\text{fin}}$ contains a finite-dimensional invariant subspace $E_{\text{fin}}$ which is defined as follows

$$E_{\text{fin}} \equiv \{P \in C_{\text{fin}} \mid P \text{ polynomial, } (3V)^r P(x, \bar{w}, z) = 0\}$$

$$\left(3 \mu = \frac{1}{2} \bar{w} q_\mu z, \quad V = \frac{\partial}{\partial x}\right).$$  \hspace{1cm} (5.1)

The general expression for the elements of $E_{\text{fin}}$ can be written as follows

$$P(x, \bar{w}, z) = \sum_{k=0}^{r-1} (x^2)^{r-k-1} h_k(x; \{\bar{w}, z, x\}),$$  \hspace{1cm} (5.2a)

where $\{\bar{w}, z, x\}$ stands for the 4-tuple

$$\{\bar{w}, z, x\} \equiv (\bar{w}, z, \bar{w}x, xz)$$  \hspace{1cm} (5.2b)
and \( h_k \) is a harmonic polynomial of degree \( k \) in the first argument and homogeneous polynomial of degrees \( 2l-n+1 \) and \( n-1 \) in \( \bar{w} \) and \( z \) respectively.

The invariance of \( E_{\nu \mu} \) follows directly from (5.2a). The invariance of the subspaces introduced below will be proved in Section 5.B.

Note that the four linear forms (5.2b) satisfy the first order differential equation

\[
(\partial_x^2) \{ \bar{w}, z, x \} = 0. \tag{5.2c}
\]

To see this we use [17]

\[
(\partial_x^{\mu})_{x\mu}(\partial_x^{\nu})_{x\nu} = 2\epsilon_{x\nu}\epsilon_{x\mu}. \tag{5.2d}
\]

The space \( C_{\nu \mu}^+ \) contains an infinite-dimensional invariant subspace

\[
F_{\nu \mu} = \left\{ f \in C_{\nu \mu}^+ \mid \exists g \in C_{\nu \mu}^+, f(x, \bar{w}, z) = \left( \frac{\partial}{\partial z} \hat{V} + \frac{\partial}{\partial \bar{w}} \right)^{\nu} g(x, \bar{w}, z) \right\}. \tag{5.3}
\]

The subspaces \( *F_{\nu \mu} = E_{\nu \mu} \) and \( F_{\nu \mu} \) \( (p = 2l+2-n) \) are orthogonal with respect to the invariant sesquilinear form (3.11) on \( C_{\nu \mu}^+ \times C_{\nu \mu}^- \). Indeed if \( P \in E_{\nu \mu}, f \in F_{\nu \mu} \) (or \( P \in E_{\nu \mu}, f \in F_{\nu \mu} \)) then

\[
L(P, f) = \int \langle P(x), f(x) \rangle dx = \int \bar{P} \left( x, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z} \right) f(x, \bar{w}, z) dx
\]

\[
= \int \bar{P} \left( x, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial \bar{w}} \hat{V} + \frac{\partial}{\partial z} \right)^{\nu} g(x, \bar{w}, z) dx
\]

\[
= (-1)^{\nu} \int \left[ \left( \frac{\partial}{\partial \bar{w}} \hat{V} + \frac{\partial}{\partial z} \right)^{\nu} \bar{P} \left( x, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z} \right) \right] g(x, \bar{w}, z) dx = 0. \tag{5.4}
\]

Conversely, if for a fixed \( f \in C_{\nu \mu}^+ \) Eq. (5.4) holds for any \( P \in E_{\nu \mu} \), then \( f \in F_{\nu \mu} \). Thus one can use (5.4) as alternative definition of \( F_{\nu \mu} \). Similarly, defining \( F_{\nu \mu} \) by (5.3), we can use (5.4) to define \( E_{\nu \mu} \).

The space \( C_{\nu \mu}^- \) contains an infinite dimensional invariant subspace

\[
F_{\nu \mu} = \left\{ f \in C_{\nu \mu}^- \mid \exists g \in C_{\nu \mu}^-, (\partial_x^2) g(x, \bar{w}, z) = f(x, \bar{w}, z) \right\}. \tag{5.5}
\]

Finally, the space \( C_{\nu \mu}^+ \) contains an infinite-dimensional invariant subspace

\[
D_{\nu \mu} = \left\{ f \in C_{\nu \mu}^+ \mid \left( \frac{\partial}{\partial z} \hat{V} + \frac{\partial}{\partial \bar{w}} \right)^{\nu} f(x, \bar{w}, z) = 0 \right\}. \tag{5.6}
\]

The subspaces \( *D_{\nu \mu} = F_{\nu \mu} \) and \( D_{\nu \mu} \) are orthogonal with respect to the invariant sesquilinear form (3.11) on \( C_{\nu \mu}^+ \times C_{\nu \mu}^+ \), i.e. for \( f_1 \in F_{\nu \mu}, f_2 \in D_{\nu \mu} \)

\[
L(f_1, f_2) = \int f_1 \left( x, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z} \right) f_2(x, \bar{w}, z) dx = 0. \tag{5.7}
\]

(The proof is analogous to that of (5.4).) It is also true that (5.7) can be used as a definition of either \( F_{\nu \mu} \) or \( D_{\nu \mu} \) depending on which of them we have defined independently.

We shall use in general the notation \( I_{\nu \mu}^{(l)} \in C_{\nu \mu}^{(l)} \) for the invariant subspaces. Note that \( *I_{\nu \mu}^{(l)} = I_{\nu \mu}^{(l)} \) \( (p = 2l+2-n) \).
5.B. Intertwining differential operators. Partial equivalence relations among the exceptional representations. We shall consider the following differential operators ($\nu = 1, 2, \ldots$; $k = 1, 2, \ldots$; $p = 1, 2, \ldots$):

\[ d^\nu: C_{x_1} \to C_{x_2}, \]  
(5.8)

\[ (d^\nu f)(x, \bar{w}, z) \equiv (\bar{w} \nabla_z)^\nu f(x, \bar{w}, z), \]
\[ \chi_1 = (m_1, m_2; c), \quad \chi_2 = (m_1 + \nu, m_2 + \nu; c + \nu); \]
\[ \partial^k: C_{x_1} \to C_{x_2}, \]  
(5.9)

\[ (\partial^k f)(x, \bar{w}, z) \equiv \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \right)^k f(x, \bar{w}, z), \]
\[ \chi_1 = (m_1, m_2; c), \quad \chi_2 = (m_1 - \nu, m_2 - \nu; c + \nu), \quad \nu \leq m_1, \nu \leq m_2; \]
\[ \partial^p: C_{x_1} \to C_{x_2}, \]  
(5.10)

\[ (\partial^p f)(x, \bar{w}, z) \equiv \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \right)^p f(x, \bar{w}, z), \]
\[ \chi_1 = (m_1, m_2; c), \quad \chi_2 = (m_1 + k, m_2 - k; c + k), \quad k \leq m_2; \]
\[ \partial^\nu p: C_{x_1} \to C_{x_2}, \]  
(5.11)

\[ (\partial^\nu p f)(x, \bar{w}, z) \equiv \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial w} \right)^p f(x, \bar{w}, z), \]
\[ \chi_1 = (m_1, m_2, c), \quad \chi_2 = (m_1 - p, m_2 + p; c + p), \quad p \leq m_1. \]

The relations between $\chi_1$ and $\chi_2$ in the above formulae follow from

\[ QT_{x_1}(g) = T_{x_2}(g)Q \quad (Q = d^\nu, \partial^k, \partial^p) \]  
(5.12)

for $g = m \in M$ and $g = a \in A$. Eqs. (5.12) are true (without restrictions on $\chi_1$) for $g = \bar{n} \in \tilde{N}$. They are established most easily in infinitesimal form. They are trivial for the generators of $\mathfrak{N}$ and $\mathfrak{M}$ (see (2.18a) and (2.18b)) and for the generator of $\mathfrak{A}$ (2.18c) it suffices to observe that

\[ [(\eta \nabla)_y, x \nabla] = r(\eta \nabla)_y, \]  
(5.13)

\[ \eta = \bar{w}qz \quad \text{or} \quad \frac{\partial}{\partial z} q^+ \frac{\partial}{\partial w} \bar{w}q e \frac{\partial}{\partial z} \frac{\partial}{\partial w} e qz \quad (\eta^2 = 0), \]

to obtain

\[ (\eta \nabla)^y D_1 = D_2(\eta \nabla)^y, \]  
(5.14)

\[ D_i = -2 - c_i - x \nabla \quad (i = 1, 2). \]

The intertwining property (5.12) is not valid for $g = n \in N$ for general $\chi_1$ in (5.8–5.11). We shall find the restrictions on $\chi_1$ brought by imposing (5.12).

First we establish the commutation formulae:
\[(\eta \nabla') x, x_\mu = r \eta r(\eta \nabla')^{-1}, \]
\[(\eta \nabla') x^2 = 2r(x \eta)(\eta \nabla')^{-1}, \]
\[ [(\eta \nabla') x, x_\mu(x \nabla)] = r [\chi_\mu, (x \nabla) + (r - 1 + x) \eta_\mu](\eta \nabla')^{-1}. \quad (5.15) \]

Using (5.15) and (2.18d) we derive
\[(\eta \nabla') C_\mu^{1c} = C_\mu^{2c}(\eta \nabla') + 2r [(1 + r + c + x \nabla) \eta_\mu - (x \eta) \nabla_\mu](\eta \nabla')^{-1}. \quad (5.16) \]

Then we obtain
\[ [\eta \nabla', \overline{w} q_k \frac{\partial}{\partial w} + z' q_k \frac{\partial}{\partial z}] = 2r(\eta_4 \nabla_k - \eta_k \nabla_4)(\eta \nabla')^{-1}, \quad (5.17a) \]
\[ [\eta \nabla', \overline{w} q_k \frac{\partial}{\partial w} - z' q_k \frac{\partial}{\partial z}] = 2r \varepsilon_{kmn} \eta_j \nabla_m(\eta \nabla')^{-1}. \quad (5.17b) \]

Using (5.17) and (2.18d) we find
\[(\overline{w} \nabla z)^\ast C_\mu^{1l} = C_\mu^{2l}(\overline{w} \nabla z)^\ast + 2r \left( \frac{m_1 + m_2}{2} - (x \nabla) \right) \eta_\mu + (x \eta) \nabla_\mu \right)(\eta \nabla')^{-1} \quad (5.18a) \]

which summed with (5.16) (for \( \eta = \overline{w} qz \) and \( r = v \)) gives
\[ d^r C_\mu^{1} = C_\mu^{2} d^r \left[ 1 + v + c + \frac{m_1 + m_2}{2} \right] d^{r-1}(\overline{w} q_z). \quad (5.18b) \]

If
\[ 1 + v + c + \frac{m_1 + m_2}{2} = 0 \quad (5.18c) \]
we obtain
\[ d^r C_\mu^{1} = C_\mu^{2} d^r. \quad (5.18d) \]

For fixed \( v \) (5.18c) means that we have two free parameters, \( c \) being real, negative and integer or half-integer. Thus we set
\[ c = -1 - v - l, \quad l = \frac{m_1 + m_2}{2} = 0, \frac{1}{2}, 1, \ldots; \quad (5.18e) \]
\[ m_2 = n - 1, \quad n = 1, 2, \ldots, 2l + 1, \quad m_1 = 2l - n + 1. \]

Thus (5.18d) is fulfilled iff (cf. (3.1))
\[ \chi_1 = \chi^m, \quad \chi_2 = \chi^{i^m}. \quad (5.18f) \]

Analogously we find that
\[ d^\ast C_\mu^{1} = C_\mu^{2} d^\ast \quad (5.19) \]
holds iff \( \chi_1 = \chi^{i^+} \) and \( \chi_2 = \chi^{i^+}; \)
\[ \partial^\ast C_\mu^{1} = C_\mu^{2} \partial^\ast \quad (5.20) \]
holds iff either \( \chi_1 = \chi^{i^m}, \ \chi_2 = \chi^{i^+} \) (cf. (3.3a)), or \( \chi_1 = \chi^{i^m} \) (cf. (3.3b)), \( \chi_2 = \chi^{i^m}, \) or
\[ \chi_1 = (m, m + k; - k/2), \ \chi_2 = \chi^{i_1}; \]
\[ \partial^\ast C_\mu^{1} = C_\mu^{2} \partial^\ast \quad (5.21) \]
holds if either \( \chi_1 = \chi_{1i}^+, \chi_2 = \chi_{1i}^-, \) or \( \chi_1 = \chi_{1i}^+, \chi_2 = \chi_{1i}^-, \) or \( \chi_1 = (m+k, m; -k/2), \chi_2 = \tilde{\chi}_1. \) In (5.19), (5.20), (5.21) the 3-tuple \((l, r, n)\) has the same ranges as in (5.18e); \( m = 0, 1, \ldots; k = 1, 2, \ldots; p = 1, 2, \ldots, 2l+1. \)

Later on we shall need all equivalence relations between the representations in (3.1) and (3.3). For this we note that making the change \( p = 2l+2-n, n = 1, 2, \ldots, 2l+1 \) in (5.21) we obtain

\[
\partial'^{2l+2-n} C_{\mu} = C_{\mu}^\alpha \partial^{2l+2-n}
\]

for \( \chi_1 = \chi_{im}, \chi_2 = \chi_{in}, \) or \( \chi_1 = \chi_{im}^+, \chi_2 = \chi_{in}^+. \)

Summarizing the results obtained so far in this subsection we have proved:

**Proposition 5.1.** Let \( d', d'', an, alp (p = 2l+2-n) \) are the differential operators \((5.8-5.11). \) Then all partial equivalence relations realized through these operators are the following

\[
d' T_{im} = T_{im} d',
\]

\[
d'' T_{im} = T_{im} d''
\]

\[
\partial'^n T_{im} = T_{im} \partial^n,
\]

\[
\partial'^n T_{im} = T_{im} \partial^n
\]

\[
\partial'^p T_{im} = T_{im} \partial^p,
\]

\[
\partial'^p T_{im} = T_{im} \partial^p
\]

\[
\partial^k T_{xi} = T_{xi} \partial^k.
\]

\[
\partial^k T_{xi} = T_{xi} \partial^k
\]

(In (5.23) \( l = 0, \frac{1}{2}, 1, \ldots; r = 1, 2, \ldots, 2l+1. \) In (5.24) \( \chi_1 = (m, m+k; -k/2), \chi_1 = (m+k, m; -k/2), \chi_1 = (m, m+k; -k/2), \chi_2 = \chi_{im}^+. \))

**Remark.** The representations \( T_{xi} \) and \( T_{zi} \) must be equivalent to \( T_{zi} \) and \( T_{zi} \) respectively in accord with our previous statement. In order to verify it we must show that the operators \( G_{zi} \) and \( G_{zi} \) reduce to \( \partial^k \) and \( \partial^k \) respectively. The proof is straightforward.

Another useful fact is:

**Proposition 5.2.** Let \( \partial'^n \) and \( \partial'^p \) be the operators (5.10-5.11) and \( F_{im}, D_{im} \) be as in (5.5-5.6). Then

\[
\partial'^n F_{im} = 0,
\]

\[
\partial'^p F_{im} = 0,
\]

\[
\partial'^n C_{im} = D_{im},
\]

\[
\partial'^p C_{im} = D_{im}.
\]

**Proof:** Let \( f \in F_{im}. \) Then there is \( g \in C_{im}: f = d'g \) and so

\[
\partial'^n f = \partial'^n d'g = \sum_{k=0}^{n} \binom{n}{k} (\partial'^{n-k} d') \partial^k g.
\]
Noting that $\partial^r g = 0 \ (m_2 = n - 1)$ and $\partial d = 0 \ (w\sqrt{\epsilon} \frac{\partial}{\partial z}) \ (w\sqrt{z}) = w\sqrt{\epsilon} w = -w\sqrt{\epsilon} w = -\tilde{w} e\tilde{w} \sqrt{\mu} \sqrt{\mu} = 0$ we conclude that every term in the above sum equals zero which gives (5.25a). For (5.25b) we use $\tilde{w}^2 \epsilon + g = 0 \ (m_1 = 2l - n + 1)$ and $\partial^d = 0$. For (5.25c) let $f \in C^{i' -}_{\text{fin}}$. We must show $d'' \epsilon d' f = 0$ which follows from $d'' f = 0 \ (m_1 = \nu - 1)$ and $d' \partial = 0$. Analogously (5.25d) follows using $d'' f = 0 \ (m_2 = \nu - 1, f \in C^{i' +}_{\text{fin}})$ and $d' \partial' = 0$. For (5.25c,d) we use also invertibility of $\partial''$ and $\partial''$ which is established in Proposition 5.7 below.

Next using Propositions 5.1 and 5.2 we shall show that $I_{\text{fin}}^{\lambda \pm}$ are indeed invariant subspaces of $C_{\text{fin}}$.

**Proposition 5.3.** Let $E_{\text{fin}}, F_{\text{fin}}, F'_{\text{fin}}$, and $D_{\text{fin}}$ be as in (5.1), (5.3), (5.5) and (5.6). Then

$$T_{\text{fin}} E_{\text{fin}} = E_{\text{fin}}, \quad T_{\text{fin}} F_{\text{fin}} = F'_{\text{fin}}, \quad T_{\text{fin}} F'_{\text{fin}} = F_{\text{fin}}, \quad T_{\text{fin}} D_{\text{fin}} = D_{\text{fin}}.$$ (5.26)

**Proof:** The invariance of $E_{\text{fin}}$ was shown in Section 5.A. To obtain the rest of (5.26) we apply (5.23a) to $C_{\text{fin}}$, (5.23b) to $C_{\text{fin}}'$, and (5.23d) to $C_{\text{fin}}''$. For the last equation besides the definitions we use also (5.25c).

For exceptional reducible representations $T^z$ and $T^z$ are only partially equivalent because the intertwining operators $G_z$ annihilate the invariant subspaces in their domains and map onto the invariant subspaces of their images. This is the content of the following

**Proposition 5.4.** Let $G_z^\pm$ is $G_z$ with normalization $n_\pm(\chi)$, set

$$G_{\text{fin}}^{(\lambda \pm)} \equiv G_{\text{fin}}^{(\lambda \pm)} f(\chi).$$ (5.27)

Then

$$\text{Ker} G_{\text{fin}}^{(\lambda \pm)} = I_{\text{fin}}^{(\lambda \pm)}, \quad \text{Im} G_{\text{fin}}^{(\lambda \pm)} = I_{\text{fin}}^{(\lambda \pm)}.$$ (5.28)

where Ker $A$ is the set of vectors mapped to the zero vector, Im $A$ is the image of the mapping $A$.

**Proof:** Note that

$$\text{Im} G_{\text{fin}} = E_{\text{fin}} \Leftrightarrow d'' G_{\text{fin}} = 0,$$ (5.29a)

$$\text{Ker} G_{\text{fin}} = F_{\text{fin}} \Leftrightarrow G_{\text{fin}} d'' = 0,$$ (5.29b)

$$\text{Im} G_{\text{fin}}^+ = F_{\text{fin}} \Leftrightarrow G_{\text{fin}} G_{\text{fin}} = 0,$$ (5.29c)

$$\text{Ker} G_{\text{fin}}^+ = E_{\text{fin}} \Leftrightarrow G_{\text{fin}} = 0,$$ (5.29d)

$$\text{Im} G_{\text{fin}}^+ = D_{\text{fin}} \Leftrightarrow d'' G_{\text{fin}}^+ = 0,$$ (5.29e)

$$\text{Ker} G_{\text{fin}}^+ = F'_{\text{fin}} \Leftrightarrow G_{\text{fin}}^+ = 0,$$ (5.29f)

$$\text{Im} G_{\text{fin}}^+ = F'_{\text{fin}} \Leftrightarrow G_{\text{fin}}^+ G_{\text{fin}} = 0,$$ (5.29g)

$$\text{Ker} G_{\text{fin}}^+ = D_{\text{fin}} \Leftrightarrow G_{\text{fin}}^+ = 0.$$ (5.29h)
The proof of (5.29) in the "⇒" direction is trivial. To prove it in the "⇐" direction we use the fact that \( I_{lm}^{\pm} \) are invariant subspaces and \( G_{lm}^{\pm} \) are intertwining operators. Note that we prove (5.29c,d) using (5.29a,b), and (5.29g,h) using (5.29e,f). Now to prove Proposition 5.4 we shall establish the validity of the equations in the right-hand sides of (5.29). (We refer to them by the respective number of the relations.)

To prove (5.29a) we note that the intertwining kernel

\[
\mathcal{G}_{lm}(x; \bar{w}_1, z_1; \bar{w}_2, z_2) = \frac{n - (\chi_{lm})(-1)^l}{(2\pi)^2} (\zeta_1 x)^{2l-n+1} (\zeta_2 x)^{n-1} \left( \frac{x^2}{2} \right)^{r-1} \quad (5.30)
\]

(\( \zeta_1 \) and \( \zeta_2 \) are given in (4.14)) is a polynomial in \( x \) of the type (5.2) and is therefore annihilated by \( d^r \). The RHS of (5.29b) is reduced to (5.29a): Let \( f \in C_{lm}^+ \)

\[
(\mathcal{G}_{lm} d^r f)(x_1, \bar{w}_1, z_1)
\]

\[
= \int \frac{G_{lm} \left( x_{12}; \bar{w}_1, z_1; \frac{\partial}{\partial \bar{w}_2} t e, \frac{\partial}{\partial z_2} \right)}{(2l-n+1)! (n-1)!} \left( \frac{\partial}{\partial z_2} \frac{\partial}{\partial \bar{w}_2} \right)^r f(x_2; \bar{w}_2, z_2) dx_2
\]

\[
= (-1)^r \int \left( \frac{\partial}{\partial \bar{w}_2} t e \right) \left( \frac{\partial}{\partial z_2} \right)^r \frac{1}{(2l-n+1)! (n-1)!} \cdot G_{lm} \left( x_{12}; \bar{w}_1, z_1; \frac{\partial}{\partial w_2} t e, \frac{\partial}{\partial z_2} \right) f(x_2; \bar{w}_2, z_2) dx_2 = 0.
\]

Using the \( p \)-space picture we see that

\[
G_{lm}^+(p) = \frac{(l + |n - l - 1|)! \left( \frac{\sqrt{2}}{2} p \right)^{2l+1-n} (p^2/2)^{l+n+1}}{(l - |n - l - 1|)! (2l + n + 1)! n!} \cdot \sum_{s=-n-1}^{l} \frac{(-1)^{l-s} (l+n+1+s)! (l+n-s)! (s+|n - l - 1|)! \Pi_{lm}^{s*}}{(s+|n - l - 1|)!} \quad (5.31)
\]

\((\Pi_{lm}^{s*} = \Pi^{(n-1,2l-n+1)s} \quad \text{— cf. (4.23)})\),

is a homogeneous function of \( p \) of degree \( 2(l+n+1) \), which annihilates the Fourier transform of polynomials \( P \in E_{lm} \) since \((\mathcal{F} P)(p, \bar{w}, z) = P(i \mathcal{V}_p, \bar{w}, z) \delta(p) \) and the degree of \( P(x, \bar{w}, z) \) does not exceed \( 2(l+n+1) \). This proves (5.29c) and (5.29d).

To prove (5.29e) we shall use the expression of \( \mathcal{G}_x \) in (4.15a) with normalization \( n_+ \):

\[
\mathcal{G}_x^+(p, \bar{w}_1, z_1; \bar{w}_2, z_2) = \left( -1 \right)^{2l} \Gamma(-c-l_2) \Gamma(c+1) \left( \frac{p^2}{2} \right)^c \cdot \sum_{k=0}^{l} \frac{m_1^{m_2}}{k!} \frac{k!}{c+1+k-l_2} \left( \mathcal{G}_x \right)^{k} (z_1 \delta z_2)^{k} (\mathcal{G}_x \mathcal{Z}_1)^{m_1-k} (\mathcal{G}_x \mathcal{Z}_2)^{m_2-k}. \quad (5.32)
\]
Thus we obtain for $d^*G^+_{\gamma}$ in the $p$-space picture:

$$
\left( \frac{\partial}{\partial z} \hat{p} \frac{\partial}{\partial \bar{w}} \right) \tilde{G}^+_{\gamma}(p; \bar{w}_1, z_1; \bar{w}_2, z_2) = \frac{(-1)^{2l_2} \Gamma(-c-l_2) \Gamma(1+c+|l_1|)}{\Gamma(|l_1| - c)} \left( \frac{p^2}{2} \right)^c \cdot \\
\cdot (\bar{w}_2 \hat{p} z_2)^m \frac{m! m! \Gamma(c-l_2+v)}{\Gamma(c-l_2)} \sum_{k=0} \frac{(\bar{w}_1 \hat{w}_2)^k (z_1 \epsilon z_2)^k (\bar{w}_1 \hat{p} z_2)^{m_1 - v - k}}{k!(m_1-v-k)! (m_2-v-k)! (1+c-l_2+k+v)} \cdot \\
\cdot (\bar{w}_2 \hat{p} z_1)^{m_2 - v - k}.
$$

Observing that

$$\lim_{x \to x^+_{im}} \frac{1}{\Gamma(c-l_2)} = 0,$$

while for $\chi = \chi^+_{im}$

$$\frac{\Gamma(-c-l_2)}{\Gamma(|l_1| - c)} = (-1)^{l+v+|l|+n+1} \frac{(1+1-|n-l-1|)!}{(2l+v+1)!},$$

we conclude that (5.33) equals zero for $\chi = \chi^+_{im}$ which gives (5.29e). Eq. (5.29f) is reduced to Eq. (5.29e).

In order to prove (5.29g) and (5.29h) we need the explicit form of $G_{\gamma}^+_{\delta}$ in $x$ space, which cannot be obtained directly by setting $\chi = \chi^+_{im}$ in (4.8) (using (4.30)) because of undefined expressions of the type

$$
\left. \right|_{x \to x^+_{im}} \frac{1}{\Gamma(0)} \left( \frac{2}{x^2} \right)^{2+k} \ (k \geq 0, \ integer).
$$

We define

$$G_{\gamma}^+_{\delta}(x) = \lim_{\delta \to 0} G_{\gamma}^+_{\delta}(x)$$

with

$$\chi^+_{\delta} \equiv (\nu+n-1, 2l+v-n+1; l+1-\delta).$$

Using (5.32) we obtain

$$G_{\gamma}^+_{\delta}(x) = \lim_{\delta \to 0} (\hat{G} \tilde{G}_{\gamma}^+_{\delta})(x) = \frac{(-1)^{l+1+|l|+n+1} (1+1+|l+1+n|)!}{2l+1(2l+v+1)!} \cdot \\
\cdot (1+1-|l+1+n|)! \sum_k \frac{(v+n-1)}{k} \left( \frac{2l+v-n+1}{k} \right) \frac{k!}{(k+1-\nu)!} \frac{(\bar{w}_1 \hat{w}_2)^k (z_1 \epsilon z_2)^k}{(\bar{w}_1 \hat{w}_2)^{l+1} (z_1 \epsilon z_2)^{l+1}} \cdot \\
\cdot (\bar{w}_2 \hat{w}_2)^{2l+v-n+1-k} (\bar{w}_1 \hat{w}_2)^{v+n-1-k} A^{k+1-\nu} \delta(x)
$$

where the distribution—theoretic identity was used [8]

$$\lim_{\delta \to 0} \left( \frac{2}{x^2} \right)^{2+k-\delta} = \frac{(2\pi)^2 A^k \delta(x)}{2k!(k+1)!},$$

and the sum in $k$ is over $v-1 \leq k \leq \min(v+n-1, 2l+v-n+1)$. 
Next we derive using (4.7b'), (4.1d), (4.30) and (4.32):

\[
\begin{align*}
&\frac{n_+ (\chi_+^0)^n_- (\chi_+^0)}{(n-2l-2+\delta)[n(2l+\nu)+2+\delta]} \\
&\frac{(2l+2+\nu-\delta)^2(l+1+|l-n+1|-\delta)\Gamma(1+2l+|l-n+1|-\delta)}{\Gamma(l-n+1)|l-1+\delta|n(n-2l-2+\delta)[n(2l+\nu)+2+\delta]},
\end{align*}
\]

(5.37)

where \( A_{\delta}^\pm \) is (analogously to \( G_{\delta}^\pm \)) \( A_{\delta} \) with normalization \( n(x) \). We note

\[
G_{\delta}^ {\pm} (x) = \lim_{\delta \to 0} G_{\delta}^ {\pm} (x)
\]

and using (5.35a) and (5.37), we obtain

\[
G_{\delta}^ {\pm} G_{\delta}^ {\pm} = \lim_{\delta \to 0} G_{\delta}^ {\pm} G_{\delta}^ {\pm} = 0.
\]

Noting that \( G_{\delta}^ {\pm} G_{\delta}^ {\pm} \) equals the same expression (5.37) we obtain also

\[
G_{\delta}^ {\pm} G_{\delta}^ {\pm} = 0.
\]

This completes the proof of Proposition 5.4.

The final result of this subsection is

**Proposition 5.5.** The following isomorphisms (and equivalences of the corresponding IR's of \( G \)) among the invariant subspaces \( I \) and the factor spaces \( G/\mathbb{I} \) for all exceptional \( \chi \) (cf. (3.1), (3.3)) hold

\[
F_{\nu} \simeq C_{\nu}^{\pm}/E_{\nu} \simeq F_{\nu}^I \simeq C_{\nu}^I/D_{\nu},
\]

\[
E_{\nu} \simeq C_{\nu}^+/F_{\nu},
\]

\[
D_{\nu} \simeq C_{\nu}^I/F_{\nu}^I \simeq C_{\nu}^I \simeq C_{\nu}^I.
\]

**Proof:** (5.38a) follows from Propositions 5.1 and 5.4; (5.38b) follows from Proposition 5.4; (5.38c) follows from Propositions 5.1, 5.2 and 5.4 assuming that \( C_{\nu}^I \) have no invariant subspaces, which will be shown in Section 5.E.

We remind that discarding the spaces \( C_{\nu}^I \) and the operators \( \partial^\nu \), considering integer \( l \) and \( n = l+1 \) we obtain from Propositions 5.1–5.5 results established in [4] for symmetric tensor representations.

Our results can be compared also with those obtained in [6] for the representations of Spin(\( N, 1 \)) (double covering of \( SO_1(N, 1) \)) for the case \( N = 5 \). The authors work in the compact picture (which we do not consider) of the elementary induced representations. They introduce in the representation space the (infinite) basis compounded by the Gel'fand–Zeitlin bases of each (finite-dimensional) representation of \( K \) contained in the elementary representation \( \mathcal{X} \). The subspaces and operators are defined in terms of this basis. For that reason the comparison with such quantities in our approach is done through the labels of the representations. We find in [6] the following results: (a) the full list of the
reducible representations (which coincide with (3.1)); (b) the matrix elements of some of the operators in Eqs. (5.23); (c) Eqs. (5.25a, c) of Proposition 5.2; (d) Proposition 5.4. We also point out some errors in ref. [6]: (a) the inverse of $\delta^p$: $C_{lm}^{r} \rightarrow C_{lm}^{r'}$ is defined by composing $G^{-1}_{lm} \delta^p$ which in fact is equal to zero ($G^{-1}_{lm} \delta^p C_{lm}^{r'} = G^{-1}_{lm} D_{lm} = 0$). As we shall show in the next subsection for this inverse we need an operator $B^+_m: D_{lm} \rightarrow C_{lm}^{r}/F_{lm}^r$; (b) Scheme 1, which would contain our Proposition 5.5 is somewhat incorrect. The true scheme for the group $G$ (and $SO^+(5, 1)$) is as follows

Fig. 1. Scheme of the intertwining operators at exceptional points

5.C. Sesquilinear forms on invariant subspaces. In this subsection we shall define invariant sesquilinear forms (ISF) on the invariant subspaces of the reducible exceptional representations (3.1). The ISF’s (4.47) are actually forms on the factor spaces $C_x/I_x$ because of Proposition 5.4.

We saw in Section 4.B that the operators $G^+_m$ (with normalization factor $n_+$ (4.30)) are not defined on the spaces $C_{lm}^{r'}$ (for $\chi = \chi_{lm}^{r''}$), and $G^+_m$ (with normalization $n_-$ (4.32)) are not defined on $C_{lm}^{r'}$ (for $\chi = \chi_{lm}^{r''}$).

On the other hand $G^+_m(x)$ differs from $G^+_m(x)$ by the numerical factor

$$\frac{n_+(\chi)}{n_-(\chi)} = \frac{(1 + c + l_2)I(1 + c + |l_1|)}{(1 - c + l_2)I(1 - c + |l_1|)}.$$  (5.39)

Note that $G^+_m$ vanishes on the invariant subspaces of $C_{lm}^{r'}$ (for $\chi = \chi_{lm}^{r''}$). We have to show that their product $G^+_m$ defines a non-vanishing (and finite) ISF on $F_{lep} \times F_{lsm}$ for $\chi = \chi_{lm}$ and on $D_{lep} \times D_{lsm}$ for $\chi = \chi_{lm}$ ($p = 2l + 2 - n$). To do that we shall use a limiting procedure.

Let, for example, $\chi_{\delta} = (2l-n+1, n-1; -l-v-1+\delta)(\lim_{\delta \to 0} \chi_{\delta} = \chi_{lm})$. Consider the
sesquilinear form

\[ B_{i\alpha}(f_1, f_2) = L_{\alpha\beta}^+(f_1, f_2) = L(f_1, G_{x_\delta}^+(x_2)) \]

\[ = \int \int \langle f_1(x_1), G_{x_\delta}^+(x_1 - x_2)f_2(x_2) \rangle \, dx_1 \, dx_2. \quad (5.40) \]

which is defined for \( \delta > 0 \). In order to find the limit of (5.40) for \( \delta \downarrow 0 \) we use the Laurent expansion of \( G_{x_\delta}^+(x) \) around the point \( \delta = 0 \):

\[ G_{x_\delta}^+(x) = \frac{(-1)^{l-n+1-l-1-\nu}}{(2l+\nu)(l+\nu+1+|l+1-n|)!(l+\nu+|l+1-n|)!} \frac{1}{\delta} G_{i\alpha}(x) + \]

\[ + \lim_{\delta \downarrow 0} \frac{d}{d\delta} \delta G_{x_\delta}^+(x). \quad (5.41) \]

The crucial point is that the pole term \( \sim 1/\delta \) in (5.41) vanishes since it is proportional to \( G_{i\alpha} \) and \( \mathcal{F}_{\alpha} \). Thus we can go to the limit \( \delta \downarrow 0 \), obtaining

\[ B_{i\alpha}^+(f_1, f_2) = \lim_{\delta \downarrow 0} B_{i\alpha}(f_1, f_2) = \lim_{\delta \downarrow 0} \frac{d}{d\delta} \delta B_{i\alpha}(f_1, f_2) \]

\[ = \int \int \langle f_1(x_1), B_{i\alpha}^+(x_1 - x_2)f_2(x_2) \rangle \, dx_1 \, dx_2. \quad (5.42) \]

where

\[ B_{i\alpha}^+(x) = \frac{(-1)^{l-n+1-l-1-\nu}}{(2l+\nu)(l+\nu+1+|l+1-n|)!(l+\nu+|l+1-n|)!} G_{i\alpha}(x) \times \]

\[ \times \left[ \ln x^2 - \psi(1+l+\nu-|l+1-n|) - \psi(1+l+\nu+|l+1-n|) + \frac{1}{\nu} \right] \quad (5.43) \]

where \( \psi(x) \) is the logarithmic derivative of \( \Gamma(x) \):

\[ \psi(x) = \frac{d}{dx} \ln \Gamma(x). \]

Note that the last three terms in the parenthesis in (5.43) give no contribution to the form (5.42) (as the \( 1/\delta \) term of \( G_{x_\delta}^+ \)).

The kernel \( B_{i\alpha}^+(x) \) is not covariant under the representation \( T_{i\alpha}^+ \) of \( G \). Under dilatations and special conformal transformations it acquires non-homogeneous terms (because of the \( \ln(x^2/2) \) term). However it is not difficult to see that the form (5.42) is invariant, since the inhomogeneous terms can be split into a sum of polynomials of \( E_{i\alpha} \) (and \( E_{\nu} \)) in \( x_2 \) (respectively \( x_1 \)) (multiplied by more complicated functions of \( x_1 \) (resp. \( x_2 \))) and therefore vanishing for \( f_1 \in F_{i\nu} \) (resp. \( f_2 \in F_{i\alpha} \)).

In general we define the ISF \( B_{i\alpha}^{(\bullet\pm)} \) on \( I_{\alpha\beta}^{(\bullet\pm)} \times I_{\alpha\beta}^{(\bullet\pm)} \) by

\[ B_{i\alpha}^{(\bullet\pm)}(f_1, f_2) \equiv \lim_{\delta \downarrow 0} \frac{d}{d\delta} \delta L_{x_\delta}^{(\bullet\pm)}(f_1, f_2) = L(f_1, B_{i\alpha}^{(\bullet\pm)}f_2) \]

\[ = \int \int \langle f_1(x_1), B_{i\alpha}^{(\bullet\pm)}(x_1 - x_2)f_2(x_2) \rangle \, dx_1 \, dx_2, \quad (5.44) \]

Note that the last three terms in the parenthesis in (5.43) give no contribution to the form (5.42) (as the \( 1/\delta \) term of \( G_{x_\delta}^+ \)).
where $\chi^{(\pm)} = (m_1, m_2; c \mp \delta)$ (\(\lim_{c \to 0} \chi^{(\pm)} = \chi^{(\pm)}_{\text{int}}\)) and

$$B^{(\pm)}_{\text{int}}(x) = \lim_{c \to 0} \frac{d}{dc} \delta \chi^{(\pm)}_{\text{int}}(x).$$

(5.45)

One finds by inspection that the forms (5.44) are finite (non-vanishing) and indeed invariant under the corresponding representations of $G$.

The explicit form of the kernels (5.45) look simpler in the $x$-space picture for $c > 0$ and in the $p$-space picture for $c < 0$. We have (discarding the terms that do not contribute to the corresponding form):

$$B_{\text{int}}(x; \bar{w}_1, z_1; \bar{w}_2, z_2) = \frac{(-1)^r (2n)!}{(l+1-n)!} (\zeta_1 x)^{2l+1-n+1} (\zeta_2 x)^{r+1-n-1},$$

(5.46)

and

$$B_{\text{int}}(p; \bar{w}_1, z_1; \bar{w}_2, z_2) = \frac{(-1)^r (2n)!}{(l+1-n)!} (\zeta^2)^{2l+1-n+1} (p/2)^{l+1-n+1},$$

(5.47)

$$B_{\text{int}}(p; \bar{w}_1, z_1; \bar{w}_2, z_2) = \frac{(l+v+|l+1-n|)!(p/2)^{l+1-n+1}}{(l+1-n)!} \times \Pi_{\text{int}},$$

(5.48)

The definition of $B_{\text{int}}^{(\pm)}$ implies the following

**Proposition 5.6.** The ISF $B_{\text{int}}^{(\pm)}$ on $\mathcal{I}^{(\pm)} \times \mathcal{I}^{(\pm)}_{\text{int}}$ is related to the ISF $L_{\chi_{\text{int}}}^{(\pm)}$ on $C^{(\pm)}_{\text{int}} / C^{(\pm)}_{\text{int}} \times C^{(\pm)}_{\text{int}} / C^{(\pm)}_{\text{int}}$ by

$$B_{\text{int}}^{(\pm)}(f_1, f_2) = L_{\chi_{\text{int}}}^{(\pm)}(F_1, F_2),$$

(5.49)

where $f_1 \in \mathcal{I}_{\text{int}}^{(\pm)}$, $f_2 \in \mathcal{I}_{\text{int}}^{(\pm)}$, $F_1 \in C^{(\pm)}_{\text{int}} / C^{(\pm)}_{\text{int}}$, $F_2 \in C^{(\pm)}_{\text{int}} / C^{(\pm)}_{\text{int}}$ (in (5.49) we take arbitrary representations of the respective cosets).

**Proof:** Using definitions (5.45) and (4.47) we obtain

$$B_{\text{int}}^{(\pm)}(f_1, f_2) = L(f_1, B_{\text{int}}^{(\pm)} f_2) = L(G_{\text{int}}^{(\pm)} F_1, B_{\text{int}}^{(\pm)} G_{\text{int}}^{(\pm)} F_2)$$

$$= L(F_1, G_{\text{int}}^{(\pm)} B_{\text{int}}^{(\pm)} G_{\text{int}}^{(\pm)} F_2) = L(F_1, G_{\text{int}}^{(\pm)} F_2) = L_{\chi_{\text{int}}}^{(\pm)}(F_1, F_2).$$

(5.49)
In the derivation of (5.49) we used that on \( I_{\nu n}^{(\tau)} \)
\[
G_{\nu n}^{(\tau)^{\pm}} B_{\nu n}^{(\tau)^{\pm}} = \lim_{\delta \to 0} \frac{d}{d\delta} G_{\nu n}^{(\tau)^{\pm}} + \delta G_{\nu n}^{(\tau)^{\pm}} = \lim_{\delta \to 0} \frac{d}{d\delta} G_{\nu n}^{(\tau)^{\pm}} G_{\nu n}^{(\tau)^{\pm}}
- \lim_{\delta \to 0} \frac{d}{d\delta} G_{\nu n}^{(\tau)^{\pm}} = \lim_{\delta \to 0} \frac{d}{d\delta} \delta G_{\nu n}^{(\tau)^{\pm}} = \lim_{\delta \to 0} \frac{d}{d\delta} \delta G_{\nu n}^{(\tau)^{\pm}} = \delta G_{\nu n}^{(\tau)^{\pm}} = \delta G_{\nu n}^{(\tau)^{\pm}}.
\]
(Note that \( (\lim_{\delta \to 0} \delta G_{\nu n}^{(\tau)^{\pm}}) I_{\nu n}^{(\tau)} \sim G_{\nu n}^{(\tau)^{\pm}} I_{\nu n}^{(\tau)} = 0 \); cf. (5.29) and, e.g. (5.41).)

Analogously we can prove
\[
B_{\nu n}^{(\tau)^{\pm}} G_{\nu n}^{(\tau)^{\pm}} = \delta G_{\nu n}^{(\tau)^{\pm}} = \delta G_{\nu n}^{(\tau)^{\pm}}.
\]

Turning to the question of unitary representations in the irreducible components
of the exceptional representations \( \chi_{\nu n}^{(\tau)^{\pm}} \) we note that such representations can only arise in the case \( \nu \) integer and \( n = l + 1 \), when the ISF \( B_{\nu n}^{(\tau)^{\pm}} = B_{\nu n}^{(\tau)^{\pm}} \) becomes a hermitian form on \( I_{\nu n}^{(\tau)^{\pm}} \). This case is discussed in detail in [4].

\section{5.D. Irreducibility of the exceptional representations \( \chi_{\nu n}^{(\tau)^{\pm}} \).}
We start the study of the representations \( \chi_{\nu n}^{(\tau)^{\pm}} \) writing down the explicit expressions for the intertwining operators \( G_{\chi_{\nu n}^{(\tau)^{\pm}}} \). This needs some consideration because substituting in (4.7d) we obtain (for \( \chi_{\nu n}^{(\tau)^{\pm}} \))
\[
G_{\chi_{\nu n}^{(\tau)^{\pm}}} \sim (\zeta x)^{2l+\nu+1}(\zeta x)^{x-1} \left( \frac{2}{x^2} \right)^{n+\nu+1}
\]
which is not \textit{a priori} well defined expression \( (n+\nu+1 \geq 3 \) and the distribution \( (2/x^2)^{2+\nu} \) is singular for \( \alpha = 0, 1, \ldots \) [8]). However, using the identity
\[
(\zeta x)^k \left( \frac{2}{x^2} \right)^{x+k} = \frac{(-1)^k I(\zeta x)}{I(\zeta x+k)} (\zeta x)^k \left( \frac{2}{x^2} \right)^x
\]
(derived for nonsingular \( \alpha \) for \( k = \nu+n, \alpha = 1 \), we obtain
\[
G_{\chi_{\nu n}^{(\tau)^{\pm}}} (x; \bar{w}_1, z_1; \bar{w}_2, z_2) = \frac{n(\chi_{\nu n}^{(\tau)^{\pm}})(-1)^{l+n}}{(2\pi)^2(n+\nu)!} (\zeta x)^2l+\nu-1(\zeta x)^x-1(\zeta x)^{n+\nu} \left( \frac{2}{x^2} \right).
\]

Analogously we find
\[
G_{\chi_{\nu n}^{(\tau)^{\pm}}} (x; \bar{w}_1, z_1; \bar{w}_2, z_2) = \frac{n(\chi_{\nu n}^{(\tau)^{\pm}})(-1)^{l+n}}{(2\pi)^2(n+\nu)!} (\zeta x)^{2l+\nu+1}(\zeta x)^x-1(\zeta x)^{2l+2+\nu-n} \frac{2}{x^2}.
\]

We do not specify the choice of the normalization constant \( n(\chi) \) because all three choices (4.30), (4.31) and (4.32) are well defined for \( \chi = \chi_{\nu n}^{(\tau)^{\pm}} \). Thus the intertwining operators \( G_{\chi_{\nu n}^{(\tau)^{\pm}}} \) are well defined (as derivatives of well defined distributions) and
\[
G_{\chi_{\nu n}^{(\tau)^{\pm}}} G_{\chi_{\nu n}^{(\tau)^{\pm}}} = \delta G_{\chi_{\nu n}^{(\tau)^{\pm}}}.
\]
which means that the representations $T^{i^+}$ and $T^{-i^+}$ are equivalent (and not just partially equivalent). However, they may still turn out to be reducible if some of the operators which intertwine $T^{i^+}$ with $T^{-i^+}$ is not invertible. We start with the following

**Proposition 5.7.** Consider the operators

$$\partial^\mu B_{lvn}^i \partial^\alpha : C_{lvn}^{i^-} \to C_{lvn}^{i^+},$$

$$\partial^\nu B_{lvn}^{i^+} \partial^\beta : C_{lvn}^{i^+} \to C_{lvn}^{i^-}.$$  

Then

$$\partial^\mu B_{lvn}^i \partial^\alpha = \frac{(-1)^{l+n-2l-1} 2^{l+n}(n-1)! (2l-n+1)! (2l+v+1)!}{(l+|l-n+1|)!(l-|l-n+1|)!} \text{id}_{C_{lvn}^{i^-}}, \quad (5.55a)$$

$$\partial^\nu B_{lvn}^{i^+} \partial^\beta = \frac{(-1)^{l+n-2l+1} 2^{l+n+2}(n+2)!}{(v-1)! (2l+v+1)! 2^{l+1}(2l-n+1)!} \text{id}_{C_{lvn}^{i^+}}. \quad (5.55b)$$

The proof is given in Appendix B.

Thus we conclude (see also [12]):

**Proposition 5.8.** There are no invariant subspaces in $C_{lvn}^{i^+}$. 

**Proof:** Eq. (5.56a) means that

$$\text{Ker} \partial^\mu \equiv \{ f \in C_{lvn}^{i^-} | \partial^\mu f = 0 \} = \{0\}$$

and that the mapping

$$\partial^\nu : C_{lvn}^{i^-}/F_{lvn} \to C_{lvn}^{i^+}$$

is onto. Similarly, Eq. (5.56b) implies that

$$\text{Ker} \partial^\nu \equiv \{ f \in C_{lvn}^{i^+} | \partial^\nu f = 0 \} = \{0\}$$

and that the mapping

$$\partial^\nu : C_{lvn}^{i^+}/F_{lvn} \to C_{lvn}^{i^-}$$

is onto.

5.E. Differential identities between sesquilinear forms for exceptional representations. The relations (5.49) between the forms $B_{lvn}^{i^+}$ and $L_{\chi_{lvn}}^\perp \perp \perp$ and relations (5.56) are not the only ones between sesquilinear forms on the irreducible components of the exceptional elementary representations. The objective of this subsection is to establish six other sets of identities of that type. In order to write these identities in a simple and symmetric form, we shall use different normalizations, specific for each identity. For instance, we note that the bilinear forms $B_{lvn}^i$ and $B_{lvn}^{-i}$ are proportional to the appropriate limits of the intertwining operator $G_{\chi}^i$ normalized according to (4.31) for $\chi \to \chi_{lvn}$ and $\chi \to \chi_{lvn}^{-i}$, respectively. This follows from (5.44) and the proportionality of the intertwining operators $G_{\chi}^i$ and $G_{\chi}^j$ for non-exceptional $\chi$.

**Proposition 5.9.** The following identities hold:

$$\partial^\nu G_{lvn}^i \partial^\alpha = a_{\nu \alpha} G_{lvn}^i,$$  

(5.59)
\[
d^\nu G^{(0)}_{l:n} \, d^\nu = a_{ln} G^{(0)}_{l:n+}, \quad \tag{5.60}
\]
\[
\partial^\nu B^{l:n}_l \, \partial^\nu = b_{ln} G^{(0)}_{l:n} \, x_{l:n}^- \quad \tag{5.61}
\]
\[
\partial^\nu G^{(0)}_{l:n} \, \partial^\nu = b_{ln} G^{(0)}_{l:n} \, x_{l:n}^+ \quad \tag{5.62}
\]
\[
\partial^\nu G^{(0)}_{l:n} \, \partial^\nu = c_{ln} G^{(0)}_{l:n} \, x_{l:n}^- \quad \tag{5.63}
\]
\[
\partial^\nu B^{l:n}_l \, \partial^\nu = c_{ln} G^{(0)}_{l:n} \, x_{l:n}^+ \quad \tag{5.64}
\]

where

\[
G^{(0)}_{l:n} \equiv \lim_{\delta \to 0} \frac{d}{d \delta} \delta G^{(0)}_{l:n}, \tag{5.65}
\]

\[
a_{ln} = \frac{(2l + \nu - n + 1)(n + \nu - 1)(l + \nu + 1 + l + 1 - n)!}{(2l - n + 1)(n - 1)(l + 1 + l + 1 - n)!}, \tag{5.66a}
\]

\[
b_{ln} = \frac{(-1)^{l-n+1}l+1+1(2l-n+2)!}{n(l + 1 + 1 + 1 + 1 + 1)!} \quad \tag{5.66b}
\]

\[
c_{ln} = \frac{(-1)^{2l+\nu+1+l-n+1}l!}{(n-1)(l + 1 + 1 + 1 + 1)!} \quad \tag{5.66c}
\]

**Proof:** Note that

\[
G_{l:n}^0 = \frac{(-1)^{l+1-n-l}(l+\nu-1-1-n)!}{\nu!} B_{l:n}^+, \quad \tag{5.67a}
\]

\[
G_{l:n}^- = G_{l:n}^0 = \frac{(2l + \nu + 2)!}{\nu!} G_{l:n}^-, \quad \tag{5.67b}
\]

\[
G_{l:n}^0 = \frac{\nu!}{(2l + 2 + \nu)!} B_{l:n}^-, \quad \tag{5.67c}
\]

\[
G_{l:n}^0 = G_{l:n}^0 = \frac{(-1)^{l-1-n}(2l + \nu + 1)!\nu!}{(l + \nu - 1 + 1 - n)!} G_{l:n}^-. \quad \tag{5.67d}
\]

We start with the proof of (5.59). Let \( f \in C_{l:n}^+ \), then

\[
(d^n G_{l:n}^0 \, d^n f)(x_1; \bar{w}_1, z_1) \quad \tag{5.68}
\]

\[
= \frac{(\bar{w}_1 \nabla_1_{z_1} x_1)^{\nu}}{(2l-n+1)!} (n-1)! \int G_{l:n}^0_{x_1} \left( x_{12}; \bar{w}_1, z_1; \frac{\partial}{\partial \bar{w}_2}, t e \frac{\partial}{\partial z_2} \right) \left( \frac{\partial}{\partial \bar{w}_2}, \partial_{\bar{w}_2} \right)^* f(x_2, \bar{w}_2, z_2) dx_2 \quad \tag{5.69}
\]

\[
= \frac{(\bar{w}_1 \nabla_1_{z_1} x_1)^{\nu}}{(2l-n+1)!} (n-1)! \int \left( \frac{\partial}{\partial \bar{w}_2}, t e \frac{\partial}{\partial z_2} \right)^* G_{l:n}^0_{x_1} \left( x_{12}; \bar{w}_1, z_1; \frac{\partial}{\partial \bar{w}_2}, t e \frac{\partial}{\partial z_2} \right) \times \quad \tag{5.70}
\]

\[
x f(x_2; \bar{w}_2, z_2) dx_2 = \frac{(\bar{w}_1 \nabla_1_{z_1} x_1)^{\nu}}{(2l-n+1)!} (n-1)! \int G_{l:n}^0_{x_1} \left( x_{12}; \bar{w}_1, z_1; \bar{w}_2, z_2 \right) \times \quad \tag{5.71}
\]

\[
x f(x_2; \bar{w}_2, z_2) dx_2 \quad \tag{5.72}
\]
Consider
\[(\overline{w}_1 \overline{v}_z)_i^* (\overline{w}_2 \overline{v}_z)_j^* G_{lmn}^0 (x; \overline{w}_1, z_1; \overline{w}_2, z_2)\]
\[= K_{lm} 2^r (\overline{\delta}_1 \overline{v})^r (\overline{\delta}_2 \overline{v})^r (\zeta, x)^{2l-n+1} (\zeta, x)^{n-1} \left( \frac{x^2}{2} \right)^{r-1} \ln \frac{x^2}{2}, \quad (5.69)\]

where
\[K_{lm} = \frac{(-1)^{l+r}(l+r+l+1-n)}{(2\pi)^2(v-1)(2l+v+1)!}\]

and we have used (4.14), (4.16a) and (5.43).

The non-log terms in (5.43) do not contribute to (5.68) and (5.69) since they are proportional to $G_{lmn}$ and by Proposition 5.4 $d^\nu G_{lmn} d^\nu = 0$.

Accounting for the fact that $(\overline{\delta}_i \zeta_j) = 0$ $(i, j = 1, 2)$ we must only apply the formula
\[ (\overline{\delta}_1 \overline{v})^r (\overline{\delta}_2 \overline{v})^r (\frac{x^2}{2})^{r-1} \ln \frac{x^2}{2} = (-1)^r v!(v-1)! \left( \frac{2}{x^2} \right)^{r+1} (\zeta, x)^{r} (\zeta, x)^{r} \quad (5.70)\]
to obtain (5.59).

Further we prove (5.61) and (5.63) using $G_{x_\delta}^r$ and $G_{x_\delta}^l$ instead of $B_{lm}^r$ and $G_{x_\delta}^0$, and going to the limit $\delta \downarrow 0$ after all differentiations are done. Then we see that (5.60), (5.62) and (5.64) are corrolaries of (5.59), (5.61) and (5.63) respectively. The necessary calculations are presented in Appendix B.

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**Appendix A. Proof of formula (4.5)**

Noting that
\[(A x A_x f)(x_1; \overline{w}_1, z) = \frac{y(x) y(x)}{2^{d+2+l} l! m!} \int_{\mathbb{R}^d} (\overline{w}_1^{x_1} \overline{w}_2^{x_2} e^{i w_2^{x_2}}) w_1^{m_1} x\]
\[\times (z_2 \overline{e}^{2x_3} x_1 x_2 z_1) f(x_3; \frac{\overline{e}}{\overline{w}_2} \varepsilon, \frac{\varepsilon}{z_2}) \left( \frac{2}{x_{12}^2} \right)^{2+e+l_2} \left( \frac{2}{x_{23}^2} \right)^{2-e+l_3} dx dx_3, \quad (A.1)\]
we shall evaluate the integral
\[I(x_1; \overline{w}_1, z_1; \overline{w}_2, z_2) \equiv \lim_{\delta \to 0} I_\delta, \quad (A.2)\]
\[I_\delta \equiv \int dx_2 \overline{w}_1^{x_{12}} \overline{w}_2^{x_{23}} e^{\overline{w}_2^{x_{23}}} (z_2 \overline{e}^{2x_3} x_1 x_2 z_1)^{m_0} \left( \frac{2}{x_{12}^2} \right)^{2+e+l_2-\delta} \left( \frac{2}{x_{23}^2} \right)^{2-e+l_3}. \]
Replacing $x_{12} = x_{13} - x_{23}$ in the first bracket and $x_{23} = x_{13} - x_{12}$ in the second and using $x+y = x^2 \varepsilon$ (cf. (1.6)) we obtain
\[I_\delta = \sum_k \sum_j \left( \begin{array}{c} m_1 \\ k \\ j \end{array} \right) \int (\overline{w}_1^{x_{13}} \overline{w}_2^{x_{23}} e^{w_2^{x_{23}}})^k (z_2 \overline{e}^{2x_3} x_1 x_2 z_1)^j (w_1 e^{w_2})^{m_1-k} \times\]
\[ x (z_2 e z_1)^{m_1-s}(-2)^{2l_1-k-j} \left( \frac{2}{x_1^2} \right)^{2+c-l_1-\delta} \left( \frac{2}{x_2^2} \right)^{2-c+l_1} dx_2 \]

\[ = \sum_k \sum_j \left( \begin{array}{c} m_1 \\ k \\ j \end{array} \right) \left( \begin{array}{c} m_2 \\ k \\ j \end{array} \right) (-2)^{2l_1-k-j} (w_1 \bar{w}_1 \bar{w}_2)^{m_1-k}(z_2 e z_1)^{m_2-s} \frac{(2\pi)^2 \Gamma(l_1+c+\delta)}{\Gamma(\delta) \Gamma(2-c+l_1+c)} \times \]

\[ \times \frac{\Gamma(2-\delta) \Gamma(c-l_1)}{\Gamma(2+c-l_1+j-\delta)} \left( \frac{2}{x_1^2} \right)^{2-\delta} \left( \frac{2}{x_2^2} \right)^{2-\delta} \left( \frac{2}{\\sqrt{x_1^2}} \right)^{j} \left( \frac{2}{\\sqrt{x_2^2}} \right)^{j} \]

(A.3)

where the differential operators \( \nabla_3^+, \nabla_1^+ \) act only inside the brackets. We have used (cf. [4]) the convolution integral in \( 2h \)-dimensional space

\[ \int \left( \frac{2}{x_1^2} \right)^{\alpha} \left( \frac{2}{x_2^2} \right)^{\beta} d^{2h}x_2 = (2\pi)^h \frac{\Gamma(h-\alpha) \Gamma(h-\beta) \Gamma(\alpha+\beta-h)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(2h-\alpha-\beta)} \left( \frac{2}{x_1^2} \right)^{\alpha+\beta-h} \]

(A.4)

for \( h = 2, \alpha = 2+c-l_1-\delta, \beta = 2-c+l_1 \).

Performing the differentiation in (A.3) we obtain

\[ I_\delta = \sum_k \sum_j \left( \begin{array}{c} m_1 \\ k \\ j \end{array} \right) \left( \begin{array}{c} m_2 \\ k \\ j \end{array} \right) 2^{2l_1-s}(2\pi)^2 (-1)^{2l_1-k-j}(w_1 \bar{w}_1 \bar{w}_2)^{m_1-s} \times \]

\[ \times (z_2 e z_1)^{m_2-s} \frac{s^{l_1} \Gamma(c-l_1) \Gamma(l_1+c+\delta) \Gamma(2-\delta+j+k-s)}{\Gamma(2+c-l_1+k) \Gamma(2+c-l_1+j-\delta) \Gamma(1+\delta)} \times \]

\[ \times \left\{ \delta \left( \frac{2}{x_1^2} \right)^{2+s-\delta} \left( 2(\bar{z}_1 x_1)(\bar{z}_2 x_1) \right)^s \right\}, \]

(A.5)

where we have used (5.2d).

Using that

\[ \lim_{s \to 0} \left\{ \delta \left( \frac{2}{x_1^2} \right)^{2+s-\delta} \left( 2(\bar{z}_1 x_1)(\bar{z}_2 x_1) \right)^s \right\} = \frac{(2\pi)^2 (\bar{z}_1 \bar{z}_2)^{\delta}(x_1)}{s+1} \]

we obtain

\[ I = 2^{2l_1}(2\pi)^4 \delta(x_1)(\bar{w}_1 \bar{w}_2)^{m_1}(z_1 e z_2)^{m_2} K \]

(A.6)

where

\[ K = \sum_k \sum_j \left( \begin{array}{c} m_1 \\ k \\ j \end{array} \right) \left( \begin{array}{c} m_2 \\ k \\ j \end{array} \right) \frac{s^{l_1} \Gamma(2+j+k-s)(-1)^j+k+s \Gamma(l_1-c) \Gamma(c-l_1)}{(j+1) \Gamma(2-c+l_1+k) \Gamma(2+c-l_1+j)}. \]

(A.7)

To evaluate \( K \) we use twice the formula [9]

\[ F(a, b, c; 1) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j) \Gamma(b+j) \Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(c+i) j!} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \]

(A.8)
and
\[ \sum_{k=0}^{m} \frac{\Gamma(\alpha+k)}{k!} = \frac{\Gamma(\alpha+m+1)}{\alpha m!}. \]  
(A.9)

Thus we have
\[ K = \frac{1}{(l_1^2 - c^2)(l_2^2 - l_1^2 - c^2)}. \]  
(A.10)

Noting that
\[ (A_x A_{x_\omega}) f(x_1; \overline{w}_1, z_1) = \frac{\gamma(\lambda) \gamma(\omega_\lambda)(-1)^{m_1+m_2}}{m_1! m_2! 2^{l_2 + 2l_1}} \int I(x_{13}; \overline{w}_1, z_1; \overline{w}_2, z_2) f \left( x_3; \frac{\partial}{\partial w_2}, \frac{\partial}{\partial z_2} \right) \]  
we end up with (4.5).

Appendix B. Proof of formulae (5.56), (5.60)-(5.64)

We start with the proof of (5.61). We reduce the evaluation of \( \partial^p B_{i n}^+ \partial^p \) to calculating
\[ \left( \frac{\partial}{\partial \overline{w}_1} \right)^p \left( \frac{\partial}{\partial \overline{z}_1} \right)^p \]  
\[ G_{k_\lambda}^+ \left( x; \overline{w}_1, z_1; \overline{w}_2, z_2 \right) \]
\[ = \left( \overline{w}_2 \gamma_\xi \frac{\partial}{\partial \overline{z}_2} \right)^p \left( -\frac{1}{2} \right)^{l+n} \frac{n_+ (\chi_\xi^-)}{(2\pi)^2} \left( \overline{w}_2 \chi z_1 \right)^{\nu + n - 1} \frac{(2l+n+1)! \Gamma(2l-n+2-\delta)}{(v-1)! \Gamma(-\delta)} \times \]
\[ \times \left( \frac{2}{\chi^2} \right)^{2l-n+3+\nu+j+\delta} \]  
\[ (\overline{w}_1 \chi z_2)^{v-1-j} (\overline{w}_2 \chi z_1)^{2l+n+1-j} \]  
(B.1)

(we have changed \( B_{i n}^+ \) with \( G_{k_\lambda}^+ \)).

For (B.1) and (B.2) we have used twice (A.8) and the formula
\[ \left( \frac{\partial}{\partial \overline{w}_1} \right)^k \left( \frac{\partial}{\partial \overline{z}_2} \right)^k \]  
\[ g(z_2) = (-2)^k \frac{n! \Gamma(2+n+\alpha)}{(n-k)! \Gamma(n+2+\alpha-k)} \left( \overline{w}_1 \chi \overline{w}_2 \right)^k \left( \overline{w}_1 \chi z_2 \right)^{n-k} g(z_2), \]  
(B.3)
where \( g(z_2) \) is a homogeneous function of \( z_2 \) of degree \( \alpha \).

Taking into account that (cf. (4.30))

\[
\lim_{\delta \downarrow 0} \frac{n^\prime(\chi^\prime) I(j - \delta)}{I(-\delta)^2} = \begin{cases} 0, & j \neq 0, \\ (-1)^{\lfloor n - l - 1 \rfloor + 1} \frac{(l + l + 1 - n)!(l - l + 1 - n)!}{(l + l + 1 - n)!l(l - l + 1 - n)!} & j = 0, \end{cases}
\]

we recover (5.61) using the definitions (5.45) and (5.53).

The proof of (5.63) is similar to that of (5.61) and we omit it.

Eqs. (5.60), (5.62) and (5.64) are direct consequences of (5.59), (5.61) and (5.63) respectively, using the intertwining properties of the operators \( G_z \) and property (5.50).

Following the lines of the derivation of (5.61) we calculate (for (5.56a))

\[
\left( \frac{\partial}{\partial \bar{w}_1} eV_{21} \right)^{2l+2-n} \left( \frac{\partial}{\partial \bar{w}_2} eV_{22} \right)^n G^z_{\omega^z} (x; \bar{w}_1, z_1, \bar{w}_2, z_2)
\]

taking the limit \( \delta \downarrow 0 \) after the differentiations and using Eq. (A.8). The same procedure is applied for (5.56b).

REFERENCES


